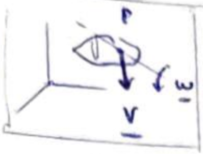
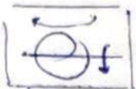
3D rotation (Morin)

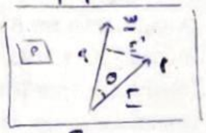
1. Preliminaries

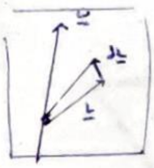
T1 

- For any point in a rigid body P, the instantaneous motion can be decomposed into
  - the translational motion of P
  - a rotation round some axis through P

e) ball on rod , now rotate the rod ; still, there's a third axis

T2
- If an object rotates with $\underline{\omega}$, the velocity of any point on the object is given as $\underline{v} = \underline{\omega} \times \underline{r}$ (more meaningfully $\frac{d\underline{r}}{dt}$)





- the same holds for any object fixed to the body frame

T3 Let S_1 rotate w $\underline{\omega}_{12}$ wrt S_2
Let S_2 rotate w $\underline{\omega}_{23}$ wrt S_3
Then S_1 rotates w $\underline{\omega}_{13}$ wrt S_3
$= \underline{\omega}_{12} + \underline{\omega}_{23}$

[All stable origins have same ω] e) (8.3)

P pick points 

$v_{P_1P_2} = \omega_{12} \times \underline{r}$

$v_{P_2P_3} = \underline{\omega}_{23} \times \underline{r}$

$v_{P_1P_3} = (\underline{\omega}_{12} + \underline{\omega}_{23}) \times \underline{r}$

holds for any P_1 of S_1 and S_3 at rest
$\Rightarrow S_1$ must rotate with $(\underline{\omega}_{12} + \underline{\omega}_{23})$ wrt S_3

e) massage a ball around

P consider the motion relative to P. imagine a sphere centered at P.

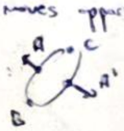
- Rigid $\Rightarrow$ it must be sent into itself

P' these two points on the sphere must remain where they are

 consider the great circle of A after an infinitesimal step

- All points move $\perp$ to the circle, otherwise distances to A change (∦ rigid)
/step ε change ε²/

- cannot move in same direction or centre will shift $\Rightarrow$ at least one point moves opposite to A



- B and C remain fixed by continuity

- this diameter remains stationary
- id'd with a rotation

2. Inertia tensor

rotation about an axis through origin

$x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$; $\omega_x\hat{\underline{x}} + \omega_y\hat{\underline{y}} + \omega_z\hat{\underline{z}}$

$$\underline{L} = \int \underline{r} \times (\underline{\omega} \times \underline{r})\, dm$$

$$\Rightarrow \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \int (y^2+z^2)\,dm & \int -xy\,dm & \int -xz\,dm \\ -\int yx\,dm & \int (z^2+x^2)\,dm & \int -yz\,dm \\ -\int zx\,dm & \int -zy\,dm & \int (x^2+y^2)\,dm \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$ /bashing/

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$ /IPhO 2018/ /Polarisation/ /normal modes/

$$\boxed{\underline{L} = \underline{\underline{I}} \cdot \underline{\omega}}$$

→ real symmetric
→ $\sum m(x^2+y^2)$ discrete, etc.
→ only in orthonormal bases (due to cross product properties)

$$\rightarrow I_{ij} \equiv \int dm \left(\delta_{ij} x_k x_k - x_i x_j\right)$$

→ obviously $\underline{L}$ depends on the origin and what the body does
→ however! the axes will change the perceived $\omega_x, \omega_y, \omega_z$, L_x, L_y, L_z, but two observers with the same origin have the same vector

e) $\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \cancel{I_{xz}\omega_3} \\ \cancel{I_{yz}\omega_3} \\ I_{zz}\omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ mr^2\omega_3 \end{pmatrix}$

e) $\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} -xz \\ -yz \\ r^2 \end{pmatrix} r^2$ [natural, freeze the system and you wouldn't know if it rotates about y or z]

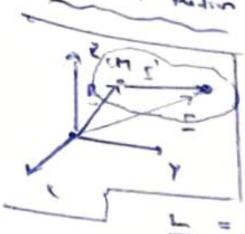
e) $\underline{L} = 2m\omega_3(0, 0, r^2)$
[I_{zx}, I_{xy} vanish due to z-axis symmetry]

e) $2m\omega b^2$
$m\omega(m a\hat{\underline{x}} + m a\hat{\underline{z}})$

$T = \int dm\, |\underline{\omega}\times\underline{r}|^2/2 \overset{|\text{both}|}{=} \frac{1}{2}\underline{\omega}\cdot\underline{\underline{I}}\,\underline{\omega} = \boxed{\frac{1}{2}\underline{\omega}\cdot\underline{L}}$ $\rightarrow \frac{I\omega^2}{2}$ if conserved

p2

General motion



use $\boxed{T4}$,
only useful expansion
is about CM

• Angular momentum (ang. vel. about CM)

$$\underline{L} = \int (\underline{R}' + \underline{r}') \times \left(\underline{V}_{cm} + (\underline{\omega}' \times \underline{r}')\right) dm$$

$$\underline{L} = \int (\underline{R}' \times \underline{V}_{cm})\, dm + \int \underline{r}' \times (\underline{\omega}' \times \underline{r}')\, dm \quad \Big/ \underbrace{\int dm\, \underline{r}' = \underline{0}}_{\text{vanish}},\ \int dm\, \underline{v}' = \underline{0} \Big/$$

$$\boxed{\underline{L} = M(\underline{R}\times\underline{V}) + \underline{L}_{cm}}$$

↓ - Parallel axis theorem

$$\underline{L} = (\underline{\underline{I}}_R + \underline{\underline{I}}_{cm})\,\underline{\omega}$$

$$\begin{pmatrix} M(Y^2+Z^2) & -MXY & -MXZ \\ -MYX & M(Z^2+X^2) & -MYZ \\ -MZX & -MZY & M(X^2+Y^2) \end{pmatrix}$$

• kinetic energy (commute with integral; operates on anything the same way)

$$T = \int \frac{1}{2} v^2\, dm$$

$$T = \int \frac{1}{2}\left(V_{cm}^2 + v'^2 + 2\underline{V}_{cm}\cdot\underline{v}'\right) dm$$

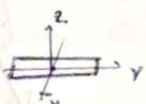
$$\boxed{T = \frac{1}{2}MV^2 + \frac{1}{2}\underline{\omega}'\cdot\underline{L}_{cm}}$$

3. Principal axes | (moduli inerziali) (moduli inerzia momento)

/ Because $\underline{\underline{I}}$ is real symmetric /

$\boxed{T4}$ For any origin there is a set of orthogonal basis vectors $1,2,3$ s.t. $\underline{\underline{I}} = \underbrace{\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}}$; i.e. s.t. $L_1 = I_1\omega_1$, $L_2 = I_2\omega_2$, $L_3 = I_3\omega_3$ (components of the ang. vel. along these axes)

→ generally obvious



$\int xy = 0$ (symm.)
but also $\int -zx = 0$, $\int -zy = 0$

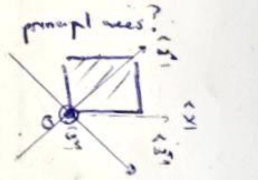
/ no proof, too much linear algebra /

• The principal axes are those round which an object can rotate with constant speed w/o any extra torque.

$\boxed{P}$ Assume an object is spinning round a fixed axis w constant speed, s.t. $\underline{\omega} = \omega_3\hat{\underline{\omega}}_3$.

if it is a principal axis, $\underline{L} = I_3\omega_3\hat{\underline{\omega}}_3$; then $\frac{d\underline{L}}{dt} = d_t(I_3\omega_3\hat{\underline{\omega}}_3) = \underline{0}$

principal axes?

$\boxed{e}$ 

→ obvious by symmetry
→ happy to spin round indefinitely

to be rigorous, show $\underline{L}$ doesn't point along rot. axis, meaning it traces out a cone around it, meaning a torque is required, ↯

• For the principal axes $\boxed{\underline{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)}$

$$\boxed{T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)}$$

⇒ connected with the body, which moves round.
But the benefits outweigh the detriments.

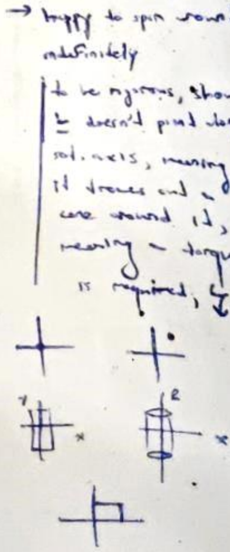
$\boxed{T5}$ If two principal moments are equal, any axis in their plane is also a principal axis

$\boxed{P}\begin{cases} I_1 = I_2 = I \\ I_3 \end{cases}$ $\underline{\underline{I}}\,\underline{u}_1 = I\underline{u}_1$ ⇒ freedom in picking the axes

$\underline{\underline{I}}\,\underline{u}_2 = I\underline{u}_2$

$\boxed{T6}$ If a flat object is symmetric wrt a rotation of $\theta \neq \pi$ in the plane, (corollary) any axis in the plane is a principal axis.

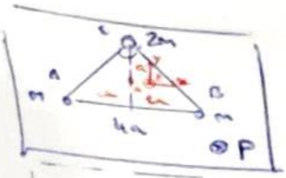
⇒ $\underline{\underline{I}}\underbrace{(a\underline{u}_1 + b\underline{u}_2)}_{\text{spans whole plane}} = I(a\underline{u}_1 + b\underline{u}_2)$

$\boxed{e}$ 

$\boxed{P}$ $\hat{\underline{\omega}}_0 \Rightarrow \hat{\underline{\omega}}_\theta$ too ⇒ span all principal

4. Examples

I | Motion after a blow



v_A, v_B, v_C ?

work in CM frame (so find CM);

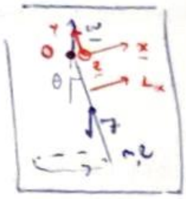
- $\underline{L} = \underline{r} \times \underline{p} = (2a, -a, 0) \times (-P, 0, 0)$
  $= aP(1, 2, 0)$
- Principal moments $\left| \begin{array}{l} I_x = 4ma^2 \\ I_y = 8ma^2 \\ I_z = I_x + I_y = 12ma^2 \end{array} \right.$
- Ang. velocities $\omega_i = \frac{L_i}{I_i} \Rightarrow \underline{\omega} = \frac{P}{4ma}(1, 1, 0)$
- Speeds relative to CM $\underline{u}_i = \underline{\omega} \times \underline{r} = \left| \begin{array}{l} (0, 0, P/4m) \\ (0, 0, -3P/4m) \\ (0, 0, P/4m) \end{array} \right.$
- Speeds in lab frame

  $\underline{v}_{cm} = -\frac{P}{4m}$

  $\Rightarrow \boxed{\underline{v}_B = \left(0, 0, -\frac{P}{m}\right)}$, rest are zero

  → intuitive
  → pivot case: use pivot as origin (and no need for last step)

II | Motion due to a torque



ω for stable motion?

- $I_x = m\ell^2/3$
  $I_y = 0$
  $I_z = m\ell^2/3$
- $L_x = I_x \omega_x = \frac{m\ell^2}{3} \sin\theta$
  $L_y = I_y \omega_y = 0$
  $L_z = I_z(\omega_z) = 0$
- $\frac{dL}{dt} = (-\hat{z}) \frac{m\ell^2}{3} \omega \sin\theta \, (\omega \cos\theta)$
- $mg\frac{\ell}{2} \sin\theta = \frac{dL}{dt}$
  directions match

  $\Rightarrow \omega = \sqrt{\frac{3g}{2\ell \cos\theta}}$
- limits

5. Euler's equations

- we want to know how the coordinates of $\underline{L}$ change in the lab frame where there are only physical torques.
- Again, stick two steps together.
  Let the angular velocity of the body in the lab frame be $\underline{\omega}$.
- $\underline{L}$ changes due to its coordinates in the body frame and due to the movement of the body frame
- Write this as $\frac{d\underline{L}}{dt} = \frac{\delta \underline{L}}{\delta t} + \underline{\omega} \times \underline{L}$ (usually zero in problems) — a vector equation which holds in any coordinates.
  Use the same frame on both sides, instantaneous 'frozen' axes

  can still write

  $\frac{d\underline{L}}{dt} = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3)$

  do this!

  $\tau_1 = \left.\frac{dL}{dt}\right|_1 = I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2$ & cyc — Euler's equations

  careful, this is interpreted as the rate of change in the body frame

READ THE EXTRA NOTE FOR CLARIFICATION

6. Free symmetric top

$\left\{ \begin{array}{l} I_1 = I_2 = I \\ I_3 \end{array} \right.$ → work in principal axes passing through CM

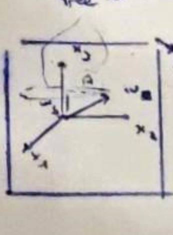
Body frame

$\left| \begin{array}{l} 0 = I\dot{\omega}_1 + (I_3 - I)\omega_3\omega_2 \\ 0 = I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 \\ 0 = I_3\dot{\omega}_3 \end{array} \right.$ $\Rightarrow \omega_3 = \text{const}$

$\left| \begin{array}{l} \dot{\omega}_1 + \left(\frac{I_3 - I}{I}\omega_3\right)\omega_2 = 0 \\ \dot{\omega}_2 + \left(\frac{I - I_3}{I}\omega_3\right)\omega_1 = 0 \end{array} \right.$ $\equiv \Omega$

$\Rightarrow \omega_1 = A\cos(\Omega t + \phi)$
$\omega_2 = A\sin(\Omega t + \phi)$

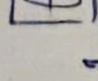
free to choose



$\Omega < 0 \Rightarrow$ just a different direction

→ this is what a guy standing on a spinning object will see! [long-exposure photograph]

→ Earth



$I_3 > I$ $\Omega \sim \frac{1}{300}\omega \Rightarrow$ period 300 d
but amplitude $(10^{-4})°$
⇒ hard to observe shift
(see 4/w3)

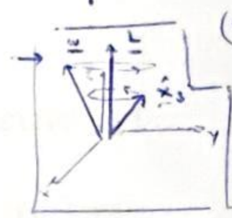
Fixed frame

- $\underline{\omega} = \omega_1\hat{\underline{x}}_1 + \omega_2\hat{\underline{x}}_2 + \omega_3\hat{\underline{x}}_3$ ⟨2⟩

→ $\underline{L} = I(\omega_1\hat{x}_1 + \omega_2\hat{x}_2) + I_3\omega_3\hat{x}_3 = I\left(\underline{\omega} + \frac{I_3 - I}{I}\omega_3\hat{\underline{x}}_3\right) = I(\underline{\omega} + \Omega\hat{\underline{x}}_3) \Rightarrow \underline{\omega} = \frac{L}{I}\hat{\underline{L}} - \Omega\hat{\underline{x}}_3$ → coplanar

in the fixed frame! $\underline{\omega} - \omega_3\hat{x}_3$ constant if no torque

(pointing $\underline{L}$)



(for $\Omega > 0$)

- Note the fixed angle and $\underline{\omega}$ and $\hat{\underline{x}}_3$, which implies that these are only able to precess around $\underline{L}$.

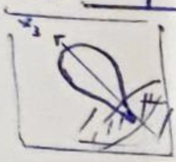
- To find the precession frequency, note $\frac{d\hat{\underline{x}}_3}{dt} = \underline{\omega} \times \hat{\underline{x}}_3 = \left(\frac{L}{I}\hat{\underline{L}} - \Omega\hat{\underline{x}}_3\right) \times \hat{\underline{x}}_3 = \left(\frac{L}{I}\hat{\underline{L}}\right) \times \hat{\underline{x}}_3$ (fixed to body)

$\tilde{\underline{\omega}} = \frac{L}{I}\hat{\underline{L}}$ (total! sideways) $\boxed{E}$ Earth $\tilde{\omega} \sim \omega_3$ which a gyro in space will see

→ $\tilde{\omega}$ and Ω are consistent, they are not different things

7. Heavy symmetric top

⇒ parts in principal axes passing through fixed point



rotation along the body axis ($\omega_3 = \dot{\psi}$)

intersection with xy plane

Euler angles θ, ϕ like in spherical polar coordinates → $\boxed{\text{READ EXTRA NOTE}}$

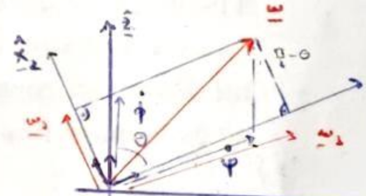
→ ang. vel. of S is $\dot{\theta}\hat{\underline{x}}_1 + \dot{\phi}\hat{\underline{z}}$ add by $\boxed{T3}$ wrt fixed frame

→ total ang. vel. of body is $\underline{\omega} = \dot{\psi}\hat{\underline{x}}_3 + \dot{\theta}\hat{\underline{x}}_1 + \dot{\phi}\hat{\underline{z}} = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\underline{x}}_3 + \dot{\phi}\sin\theta\,\hat{\underline{x}}_2 + \dot{\theta}\hat{\underline{x}}_1$

⟨1a⟩ ⟨1b⟩ ($= \dot{\beta}$)

decompose $\hat{\underline{z}}$ along $\hat{\underline{x}}_2, \hat{\underline{x}}_3$

(Interdependencies and Diagram) (for $\dot{\theta} = 0$)



$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$

$\omega_2 = \dot{\phi}\sin\theta$

Torque method

$\underline{L} = I_3\dot{\beta}\hat{\underline{x}}_3 + I\dot{\phi}\sin\theta\,\hat{\underline{x}}_2 + I\dot{\theta}\hat{\underline{x}}_1$ all change $\left| \frac{d\hat{\underline{x}}_3}{dt} = -\dot{\theta}\hat{\underline{x}}_2 + \dot{\phi}\sin\theta\,\hat{\underline{x}}_1 \right.$ etc. $\left. \right|$

$\frac{d\underline{L}}{dt} = I_3\ddot{\beta}\hat{\underline{x}}_3$
$+ (I\ddot{\phi}\sin\theta + 2I\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\beta}\dot{\theta})\hat{\underline{x}}_2$
$+ (I\ddot{\theta} - I\dot{\phi}^2\sin\theta\cos\theta + I_3\dot{\beta}\dot{\phi}\sin\theta)\hat{\underline{x}}_1$ } =

$\underline{\tau} = Mgl\sin\theta\,\hat{\underline{x}}_1$

→ $\dot{\phi}$ is the precession frequency of the top about $\hat{z}$

obvious, but here's a proof $\frac{d\hat{\underline{x}}_3}{dt}$ (fixed) $= \underline{\omega} \times \hat{\underline{x}}_3 \overset{(1)}{=} (\Omega\hat{\underline{z}}) \times \hat{\underline{x}}_3$ (however complicated it is!)

$$\Rightarrow \left| \begin{array}{ll} \dot{\beta} = \omega_3 = \text{const} & \langle 3\rangle \\ I\ddot{\phi}\sin\theta + \dot{\theta}(2I\dot{\phi}\cos\theta - I_3\omega_3) = 0 & \langle 4\rangle \\ (Mgl + I\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi})\sin\theta = I\ddot{\theta} & \langle 5\rangle \end{array} \right.$$

→ $\dot{\psi}$ is the rotation frequency about $\hat{x}_3$; only rotation if you remove the precession

Gyroscope $\dot{\theta} = 0$ (i.e. if set up to precess)

⟨4⟩ $\dot{\phi} = \Omega = \text{const}$

$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mgl = 0 \Rightarrow \Omega_\pm = \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \frac{4MIgl\cos\theta}{I_3^2\omega_3^2}}\right)$

fast - precession freq.
slow - precession freq.

→ $\Omega_- \approx \frac{Mgl}{I_3\omega_3}$
→ horizontal gyroscopes are easiest
→ high ω_3 stabilises

[pS]

Nutation

- ω_3 large (i.e. very stable)
- $\dot\theta$ small
- $\dot\phi$ small (slow-precession)

$$\Rightarrow \left| \begin{array}{l} I\dot\phi \sin\theta - I_3\omega_3\dot\theta = 0 \\ (Mgl - I_3\omega_3\dot\phi)\sin\theta + I\ddot\theta \end{array}\right. \xrightarrow{d/dt}$$

$$\dot\theta = \frac{I\sin\theta}{I_3\omega_3}\dot\phi \quad \langle 6 \rangle$$

$$\ddot\theta = \frac{I\sin\theta}{I_3\omega_3}\frac{d^2(\phi)}{dt^2}$$

$$\frac{I^2}{I_3\omega_3}\frac{d^2\phi}{dt^2} + I_3\omega_3\dot\phi = Mgl$$

$$\boxed{\omega_n = \frac{I_3\omega_3}{I}}$$

$$\frac{d^2\phi}{dt^2} + \underbrace{\left(\frac{I_3\omega_3}{I}\right)^2}_{\omega_n^2}\dot\phi = \frac{Mgl\, I_3\omega_3}{I^2}$$

$$= \underbrace{\frac{Mgl}{I_3\omega_3}}_{\Omega_s = \text{const}}\omega_n^2$$

$$\frac{d^2(\dot\phi - \Omega_s)}{dt^2} + \omega_n^2(\dot\phi - \Omega_s) = 0$$

$$\dot\phi = \Omega_s + A\cos(\omega_n t + \delta)$$

$$\phi = \Omega_s t + \frac{A}{\omega_n}\sin(\omega_n t + \delta) + \cancel{C} \quad (\text{set } 0)$$

$$\langle 6 \rangle \quad \dot\theta = -\left|\frac{I\sin\theta}{I_3\omega_3}\right| A\omega_n \sin(\omega_n t + \delta) = -\underset{\text{assumption}}{A}\underset{\sin\theta_0}{\sin\theta}\sin(\omega_n t + \delta)$$

$$\rightarrow \theta(t) = B + \frac{A}{\omega_n}\sin\theta_0\cos(\omega_n t + \delta)$$

→ larger ω_3 ⇒ larger ω_n, and more stable

Maths

- $\underline{a} \times \underline{b} = \begin{vmatrix} \hat x & \hat y & \hat z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- $\begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix}$ just in table
  det...
- Vectors are vectors

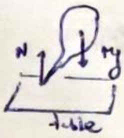
Personal notes

$\boxed{e}$ Morin (8) principal axis & $\boxed{TS}$ (rice cylinder)

$\boxed{e}$ Morin (11)

not pivoted

$\boxed{e}$ Morin (13) → CM always legal!! may

- 

table

take everything wrt CM now; pivot accelerates ⇒ not allowed; $N = Mg$ & torque $Mgl\sin\theta$ happens to be the same; only modify $I_1 = I_2 \equiv I$ [Now they are $I_{CM} = I - Ml^2$]

- $\theta \to 0$ special cases often give surprising results

Personal notes

$\boxed{e}$ (19)

$\boxed{e}$ (20)

} always figure out $\underline\omega$ first

- $\underline{L} = I\underline{\omega}$ if any axis is a principal axis

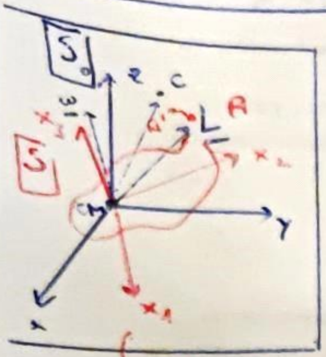
- If there is a symmetry axis
  $\underline\omega = \underline\omega_{\parallel} + \underline\omega_{\perp}$ (parallel to axis)
  $\underline{L}_{||} = \underbrace{I_{\parallel}\underline\omega_{\parallel} + I_{\perp}\underline\omega_{\perp}}_{\text{identical}}$

- You can use rotating frames to find the lab frame angular velocity.
  (Fictitious forces and torques) ← But do not use them for dynamics.

$$\boxed{\underline{L} = M\underline{r}_{CM} \times \underline{v}_{CM} + I_{\parallel}\underline\omega_{\parallel} + I_{\perp}\underline\omega_{\perp}}$$

$$\boxed{K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{\parallel}\omega_{\parallel}^2 + \frac{1}{2}I_{\perp}\omega_{\perp}^2}$$

Extra note on Euler's equations



principal axes which define the body frame because they're affixed to the body

- Remember that the motion of the body at any instant is just a pure rotation about some axis $\underline{\omega}$ (which will change) (in the lab frame)

- In the lab frame the body has some vectors $\underline{\omega}$ and $\underline{L}$.
  In the body frame the body obviously has no ω and $\underline{L}$ of its own as it is static there. However the physical (lab) $\underline{\omega}$ and $\underline{L}$ still have some projections along x_1, x_2, x_3.

- we seek to find $\frac{d\underline{L}}{dt}$ $(=\underline{\tau})$ in the lab frame: no inertial torques there, so it will be easy to match $\frac{d\underline{L}}{dt}$ to the external torques.

(However, we'll need the projection of this $\frac{d\underline{L}}{dt}$ along x_1, x_2, x_3 anyway. Always better to work along x_1, x_2, x_3 if possible. ⟨1⟩)

- in theory you could just write $\frac{d\underline{L}}{dt} = \frac{d}{dt}\left(I_1\omega_1\hat{\underline{x}}_1 + I_2\omega_2\hat{\underline{x}}_2 + I_3\omega_3\hat{\underline{x}}_3\right)$ and bash it out. However, this will involve the derivatives $\frac{d\hat{x}_1}{dt}$ $\left(= \underline{\omega}\times\hat{\underline{x}}_1 \text{ by } \boxed{T2}\right)$, so we could save ourselves the effort with a trick.

you'll need to expand this out with a determinant etc.

- Let the endpoint of $\underline{L}$ lie at some fixed point in S called A
- After time dt A ends up at position A' because S rotates as a whole wrt S_0.
  The endpoint of $\underline{L}$ is now at C. (see diagram)
- The total shift $d\underline{L}$ can be decomposed as the shift of C wrt A (i.e. $\vec{A'C}$) and the shift of A itself

$\Rightarrow$ The total velocity $\frac{d\underline{L}}{dt}$ is due the velocity of C wrt A $\frac{\delta \underline{L}}{\delta t}$ (i.e. wrt body frame and its axes x_1, x_2, x_3) and the velocity of A itself $\underline{V_A}$

A is affixed to S, so this is $\underline{\omega}\times\underline{L}$ by $\boxed{T2}$

distance from CM to A

$$\Rightarrow \frac{d\underline{L}}{dt} = \frac{\delta \underline{L}}{\delta t} + \underline{\omega}\times\underline{L}$$

[and the components here will be taken about x_1, x_2, x_3, as mentioned in ⟨1⟩]

$\frac{\delta \underline{L}}{\delta t}$ is wrt basis (x_1, x_2, x_3), not (x, y, z). 7

So this is simply $(I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3)$

$$\frac{d\underline{L}}{dt} = (I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

$$\hookrightarrow \underbrace{\begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}}$$

gives e.g. $\hat{\underline{x}}_1 (\omega_2 I_3 \omega_3 - I_2 \omega_2 \omega_3)$ etc.

$$\Rightarrow \left.\frac{d\underline{L}}{dt}\right|_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 \quad \text{\& cyclic}$$

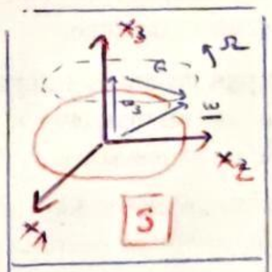
Extra note on free symmetric top

- Solve Euler's eqns for no torque $\Rightarrow$ $\omega_3 = \text{const}$, $\omega_1 = A\cos(\Omega t + \phi)$, $\omega_2 = A\sin(\Omega t + \phi)$

  ↳ this is how the lab $\underline{\omega}$'s projections on $\hat{\underline{x}}_1, \hat{\underline{x}}_2, \hat{\underline{x}}_3$ will look

- More useful to check how $\underline{\omega}$ will move around in the lab frame. Align the z-axis of the lab frame with $\underline{L}$, which is constant for a free symmetric top*.

$\Rightarrow$ in body frame (doesn't move here):



$$\Omega = \frac{I_3 - I}{I}\omega_3 = \text{const}$$

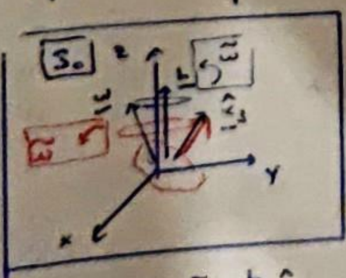
$\boxed{c}$ This makes it obvious that the Earth's rotation axis doesn't always pierce it at the same physical point $\Rightarrow$ north pole will move around the Earth's x_3 with a period of $2\pi/\Omega$

for the lab $\underline{\omega}$ and $\underline{L}$:

$$\left.\begin{aligned} \underline{\omega} &= \omega_1\hat{\underline{x}}_1 + \omega_2\hat{\underline{x}}_2 + \omega_3\hat{\underline{x}}_3 \\ \underline{L} &= I(\omega_1\hat{\underline{x}}_1 + \omega_2\hat{\underline{x}}_2) + I_3\omega_3\hat{\underline{x}}_3 \end{aligned}\right\} \Rightarrow \underline{\omega} = \underbrace{\frac{L}{I}\hat{\underline{L}} - \Omega\hat{\underline{x}}_3}_{\text{constants}}$$

linear relation $\Rightarrow$ coplanar and in a fixed configuration and $\hat{\underline{L}}$ (i.e. $\hat{\underline{z}}$)

$\Rightarrow$ $\underline{\omega}$ and $\hat{\underline{x}}_3$ can ONLY precess about $\hat{\underline{z}}$ (with some frequency $\tilde{\omega}\hat{\underline{z}}$)



turns out $\tilde{\omega} = \frac{L}{I}\hat{\underline{L}}$.

a guy in space sees the Earth precess (with its $\hat{\underline{x}}_3$) about $\hat{\underline{L}}$.

---

* - the following tricks don't work for the heavy symm. top, for then $\underline{L} \neq \text{const}$. You'll have to introduce Euler angles and bash it out.

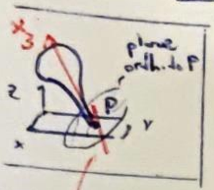
Extra note on heavy symmetric top (Euler angles)

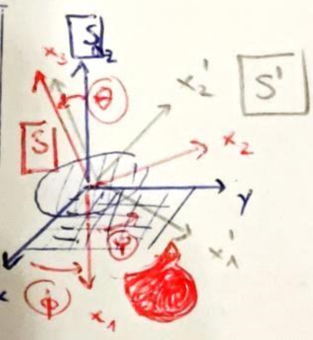
- No conservation laws $\Rightarrow$ you're doomed unless you solve it generally.
- This means you need to parametrise $\underline{\omega}$ in terms of some angular coordinates

↳ Euler angles θ, ϕ, ψ

- Introduce S_0 (lab frame), S, S' (body frame)

(obviously defined using some principal axes)





the plane intersects (xy) at x_1 which is a principal axis (since it is a part of ...)

- S is some frame with 3 principal axes, but is not affixed to the body. It is instead defined by the intersection x_1 and x_3.

- The body frame is S'; with axes (x_1', x_2', x_3) (in the air)

  S' is rotated by $\boxed{\psi}$ wrt S along

(same as before)

  S itself is related to S_0 by $\boxed{\theta}$ and $\boxed{\phi}$

- The total $\underline{\omega}$ for S' wrt S_0 can be built up

as $\underline{\omega} = \underline{\omega}_{s's} + \underline{\omega}_{ss_0}$ by $\boxed{T3}$

these are just easier to imagine

- $\underline{\omega}_{s's} = \dot{\psi}\,\hat{\underline{x}}_3$ (obvious)

- $\underline{\omega}_{ss_0} = \underbrace{\dot{\theta}\,\hat{\underline{x}}_1}_{\text{from some rotation which changes } \theta} + \underbrace{\dot{\phi}\,\hat{\underline{z}}}_{\text{from some rotation which changes } \phi}$

(this will be about $\hat{\underline{x}}_1$) (this will be about $\hat{\underline{z}}$)

this spans all scenarios

- In total, $\underline{\omega} = \dot{\psi}\hat{\underline{x}}_3 + \dot{\theta}\hat{\underline{x}}_1 + \dot{\phi}\hat{\underline{z}}$ project into $\hat{\underline{x}}_2, \hat{\underline{x}}_3$

(no projection along $\hat{\underline{x}}_1$ because we chose it horizontal ☺)

$$\underline{\omega} = \underbrace{(\dot{\psi} + \dot{\phi}\cos\theta)}_{\equiv \dot{\beta}}\hat{\underline{x}}_3 + \dot{\phi}\sin\theta\,\hat{\underline{x}}_2 + \dot{\theta}\,\hat{\underline{x}}_1$$

- θ, ϕ, ψ are simply convenient; if $\dot{\theta} = 0$: (gyroscope)

→ e.g. $\dot{\phi}$ is just the precession frequency about $\hat{\underline{z}}$

→ $\dot{\psi}$ is the rotation frequency about $\hat{\underline{x}}_3$. This is the only motion left if you remove the precession.