Short Exam 1

Problem. Collisions. A ball of mass M moves with velocity V_0 on a smooth horizontal surface. It collides elastically with a second ball of mass $m = 2\,\text{kg}$. The second ball is connected to a third ball with a relaxed massless spring $k = 1\,\text{N/m}$. The spring is long enough that the second and the third balls do not collide (Figure 1).

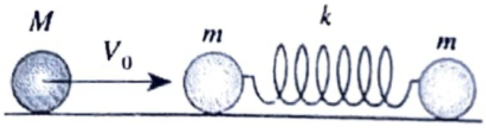


Figure 1

(a) Find the minimum mass of the first ball M_{min} for which it will collide with the second ball again.

(b) Find the time between the two collisions τ in that case.

First solve the problem approximately. If you have enough time, try to improve on your result using numerical methods.

Solution. The surface is smooth, meaning there is no friction in the system and the balls simply slide. Consider the first collision of M and m. Since it is instantaneous, the balls don't get to move during the collision, so the spring doesn't contract and the force it applies on m is negligible. In a sense, it behaves like a slack string, not a rod. Now, the collision is elastic, so we can apply both conservation of energy and conservation of momentum in finding the velocities v_1 and v_2 after the collision:

$$MV_0^2 = Mv_1^2 + mv_2^2,$$
$$MV_0 = Mv_1 + mv_2.$$

It's easiest to solve these by bringing the v_1 terms to the left hand side, using $V_0^2 - v_1^2 = (V_0 - v_1)(V_0 + v_1)$, and then dividing the two equations. We notice that $V_0 + v_1 = v_2$, and therefore

$$v_1 = \frac{M-m}{M+m}V_0, \qquad v_2 = \frac{2M}{M+m}V_0.$$

Denote by $x = 0$ the position at which the balls first collide (at time $t = 0$). Since there are no external forces on M after the collision, the coordinate of its right end will be given by $x_M(t) = v_1 t$. We now need to find an equation for the position $x_m(t)$ of the left end of m, and compare this with $x_M(t)$. The system consisting of the two balls m and the spring isn't subject to external forces, so the velocity of its centre of mass (CM) is constant and equal to $v_{\text{CM}} = \frac{mv_2}{m+m} = \frac{v_2}{2}$. Let us switch to the inertial frame in which the CM is at rest. The initial velocity of the left m in this frame is $v_2 - v_{\text{CM}} = \frac{v_2}{2}$. We will track the coordinate x' of the left end of the mass in the CM frame, taking $x'(0) = 0$ just like in the lab frame.

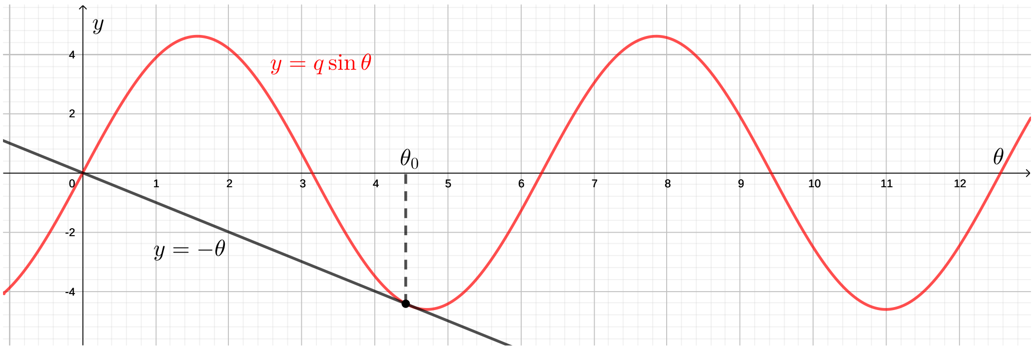
Since the CM is at rest, whenever the left m is displaced by x', the right m should be displaced by $-x'$, and the total contraction of the spring is then $2x'$. The equation of motion for the left m is thus $m\ddot{x}' + 2kx' = 0$, which is solved by $x' = A\sin\left(\sqrt{\frac{2k}{m}}t + \varphi\right)$ for some A and φ. After applying the boundary conditions $x'(0) = 0$ and $\dot{x}'(0) = \frac{v_2}{2}$, we get $x' = \left(\frac{v_2}{2}\sqrt{\frac{m}{2k}}\right)\sin\left(\sqrt{\frac{2k}{m}}t\right)$. The position of the CM frame's origin in the lab frame is simply $x_{\text{CM}} = v_{\text{CM}}t$, so upon switching back to the lab frame we find

$$x_m = \frac{v_2 t}{2} + \frac{v_2}{2}\sqrt{\frac{m}{2k}}\sin\left(\sqrt{\frac{2k}{m}}t\right).$$

Let the difference in position between the right end of M and the left end of m be $\Delta x = x_M - x_m$. To get a second collision, we require that $\Delta x \geq 0$ for some $t > 0$. Denote $M/m \equiv q$. After some algebra, our inequality reduces to

$$\left(\sqrt{\frac{2k}{m}}t\right) + q\sin\left(\sqrt{\frac{2k}{m}}t\right) \leq 0.$$

We might as well work with $\theta \equiv \sqrt{\frac{2k}{m}}t$ now. We need to find the smallest q for which the equality can be satisfied at some $\theta > 0$. For larger q there might be lots of θ that work, but for the smallest value q_{min} this should barely be possible. In other words, there should only be one $\theta = \theta_0$ at which $\theta_0 + q_{\text{min}}\sin\theta_0 = 0$, and all other θ should yield $\theta + q_{\text{min}}\sin\theta > 0$. Here's what this looks like on a graph:



The problem now reduces to finding a q_{min} such that $y = -\theta$ is a tangent to $y = q_{\text{min}}\sin\theta$ only once for all $\theta > 0$. This happens exactly at θ_0, which should lie between π and $\frac{3}{2}\pi$. The equation of the tangent at θ_0 is

$$y = q_{\text{min}}\sin\theta_0 + q_{\text{min}}\cos\theta_0(\theta - \theta_0).$$

Setting this to $y = -\theta$ gives us the following set of equations:

$$\tan\theta_0 = \theta_0, \qquad q_{\text{min}}\cos\theta_0 = -1.$$

Solving the former equation numerically, we obtain $\theta_0 = 4.494$, and it follows that $q_{\text{min}} = 4.615$. Our final answers are then

$$\boxed{M_{\text{min}} = 4.615m = 9.230\,\text{kg}} \quad \text{and} \quad \boxed{\tau = 4.494\sqrt{\frac{m}{2k}} = 4.494\,\text{s}.}$$

An approximate solution could have been found by estimating $\theta_0 = 4.5$ from the graph. This would give us $M_{\text{min}} = 9.5\,\text{kg}$ and $\tau = 4.5\,\text{s}$, which is quite good.

Short Exam 2

Problem. Betatron. The betatron is a compact particle accelerator for electrons which can bring them to relativistic velocities. It consists of two coaxial coils placed symmetrically about a thin cylindrical vacuum chamber (Figure 2). In the chamber there is a small electron source which emits electrons with zero initial velocity.

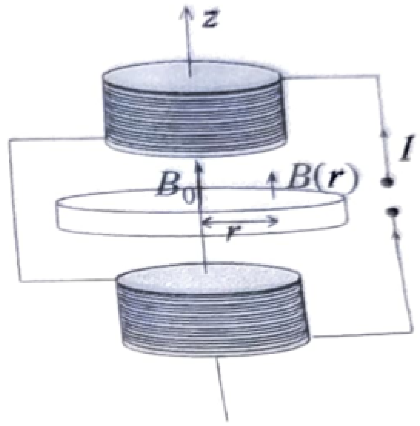


Figure 2

The magnetic field in the plane of the chamber is parallel to the z-axis and varies with the distance r to the coils' axis as follows:

$$B(r) = B_0\left(1 - \left(\frac{r}{a}\right)^2\right),$$

where B_0 is the field at the centre of the chamber and a is a constant which depends on the size of the coils and the distance between them. After the betatron is turned on, the current in the coils increases, and B_0 grows from zero to some fixed maximum value $B_{\max}$.

(a) The source is placed at a distance r_S from the z-axis so that the electrons it emits are accelerated in the chamber along circular trajectories. Find r_S/a.

(b) Find the maximum kinetic energy of the electrons $K_{\max}$ for $B_{\max} = 1.00\,\mathrm{T}$ and $a = 0.10\,\mathrm{m}$. Express your answer in eV.

Solution. (a) We will assume that the magnetic field is switched on almost immediately. In that time, the electrons will acquire some momentum, but their displacement from the source will be negligible. We will try to find r_S such that the trajectories of the electrons after reaching $B_{\max}$ are circular. Even when we're working in the relativistic regime, the total force on any electron is still $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$, where $\mathbf{p}$ is its momentum. Let us denote the unit vector along $\mathbf{p}$ by $\hat{\mathbf{n}}$, so that $\mathbf{p} = p\hat{\mathbf{n}}$. We can then write

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \hat{\mathbf{n}}\frac{\mathrm{d}p}{\mathrm{d}t} + p\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}t}.$$

Any magnetic forces are perpendicular to $\hat{\mathbf{n}}$, so those will correspond to the second term. For the case of circular motion with angular frequency $\boldsymbol{\omega}$, we can show that $\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\mathbf{n}}$. These two vectors are orthogonal, so the magnitude of the second term is just equal to ωp. After $B_{\max}$ has been reached, the electron experiences only a radial magnetic force with magnitude $evB(r_S) = e\omega r_S B(r_S)$[1]. Matching this to ωp, we find that the momentum of the electron should obey

$$p = eB(r_S)r_S = eB_{\max}\left(1 - \left(\frac{r_S}{a}\right)^2\right)r_S.$$

Now we will check how the momentum p is acquired in the first place. When the magnetic field is turned on, the changing magnetic flux Φ gives rise to an electric field according to Faraday's law, $E(2\pi r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$. At any instant the electron is subject to a force $-eE$. The total momentum it gains can be found by integrating this force with respect to time. It follows that $p = \frac{e}{2\pi r}\Phi_f$, where Φ_f is the flux at the end of the process. This flux can be found through

$$\mathrm{d}\Phi_f = B_{\max}\left(1 - \left(\frac{r}{a}\right)^2\right)2\pi r\,\mathrm{d}r \quad\Rightarrow\quad \Phi_f(r) = 2\pi B_{\max}\left(\frac{r^2}{2} - \frac{r^4}{4a^2}\right).$$

[1]Note that the formula for the Lorentz force stays the same in relativistic dynamics, provided that you interpret it as 'what you substitute in the left hand side of $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$'.

After equating our expressions for p, at $r = r_S$ we get

$$eB_{\max}\left(1-\left(\frac{r_S}{a}\right)^2\right)r_S = eB_{\max}\left(\frac{1}{2}-\frac{1}{4}\left(\frac{r_S}{a}\right)^2\right)r_S \quad \Rightarrow \quad \boxed{\frac{r_S}{a}=\sqrt{\frac{2}{3}}.}$$

(b) Using this result back in our expression for the momentum, we obtain $p = \sqrt{\frac{2}{27}}eB_{\max}a$. The kinetic energy of the electrons is given by their total energy minus their rest energy, so

$$K_{\max} = \sqrt{(m_e c^2)^2 + (pc)^2} - m_e c^2 = 1.23 \times 10^{-12}\,\mathrm{J} = \boxed{7.67\,\mathrm{MeV}.}$$

Short Exam 3

Problem. Yukawa potential. The interaction energy between a proton and a neutron in the nucleus depends on the distance r between them and is given by

$$U(r) = -U_0\frac{e^{-r/r_0}}{r/r_0},$$

where $r_0 = 1.3 \times 10^{-15}\,\mathrm{m}$. This dependence was proposed by Hideki Yukawa in 1935. For certain values of U_0 the proton and the neutron are in a bound state, forming a deuteron. The experimental value for the binding energy of the proton and the neutron in a deuteron is $\varepsilon = 2.225\,\mathrm{MeV}$. Using the Heisenberg uncertainty principle, estimate:

(a) The minimum U_0 that allows for a deuteron to form.

(b) The size of the deuteron at the given binding energy.

(c) The value of U_0 at the given binding energy.

Solution. (a) The system consists of two particles, each with mass $M \approx m_p$ and momentum p in the centre-of-mass frame. The total energy in this frame is

$$E = 2 \cdot \frac{p^2}{2M} - U_0\frac{e^{-r/r_0}}{r/r_0}.$$

This is essentially the binding energy of the system. The system is in a bound state when $E < 0$. We are given that $E = -\varepsilon = -2.225\,\mathrm{MeV}$, but we won't be making use of this just yet.

In order to relate p and r, we will apply the Heisenberg uncertainty principle. We interpret p as the uncertainty in momentum and r as the uncertainty in position, and then we can write $pr \sim \hbar$, giving us

$$E = \frac{\hbar^2}{Mr^2} - U_0\frac{e^{-r/r_0}}{r/r_0},$$

or, after setting $\rho \equiv r/r_0$,

$$E = \frac{\hbar^2}{Mr_0^2}\cdot\frac{1}{\rho^2} - U_0\frac{e^{-\rho}}{\rho}.$$

No matter what U_0 actually is, the system will assume a ρ which minimises the energy. This means that ρ should satisfy $\frac{\mathrm{d}E}{\mathrm{d}\rho} = 0$, i.e.

$$\frac{2\hbar^2}{Mr_0^2} = U_0\rho(1+\rho)e^{-\rho}. \qquad (1)$$

We can substitute this condition into our expression for E to find that

$$E = U_0 \frac{(\rho - 1)e^{-\rho}}{2\rho}.$$

Since bound states have $E < 0$, we conclude that $\rho < 1$. Let us now return to finding the minimum U_0. Referring to (1), this occurs when $f(\rho) = \rho(1+\rho)e^{-\rho}$ is maximised. The derivative of this function is $f'(\rho) = (-\rho^2 + \rho + 1)e^{-\rho}$, with zeroes at $\rho = \frac{\pm\sqrt{5}-1}{2}$. We see that the derivative is negative for all $\rho > 1$. Thus the maximum value of $f(\rho)$ for a bound state is realised precisely at $\rho = 1$, where $f(1) = 2e$. Going back to (1), we find

$$\boxed{U_0 = \frac{e\hbar^2}{Mr_0^2} = 66.9\,\text{MeV}.}$$

(b) Now we account for the actual experimental data on E. We have

$$-\varepsilon = U_0 \frac{(\rho - 1)e^{-\rho}}{2\rho},$$

and (1) is still in effect. We divide the two to find

$$\frac{\hbar^2}{Mr_0^2\varepsilon} = \frac{\rho^2(1+\rho)}{1-\rho}.$$

The left hand side is equal to 11.06, and we need the respective ρ. Solving this numerically, we get $\rho = 0.87$, which corresponds to $r = 0.87 r_0 = \boxed{1.1 \times 10^{-15}\,\text{m.}}$

(c) Going back to the expression for the total energy with our value for ρ, we calculate

$$\boxed{U_0 = 71.1\,\text{MeV}.}$$

Theoretical Exam

Problem 1. Uniform motion. Two point masses move uniformly with velocities $\mathbf{v_1}$ and $\mathbf{v_2}$ respectively. At time $t = 0$ their positions are $\mathbf{r_{01}}$ and $\mathbf{r_{02}}$.

(a) Find a formula for the time t_{min} when the particles are closest to each other.

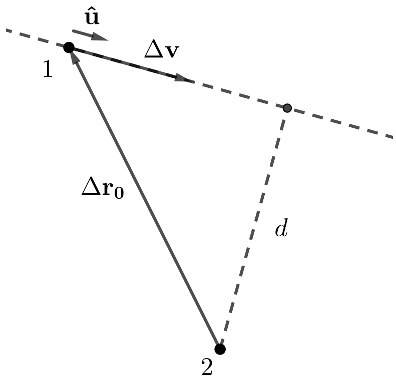
(b) Find a formula for this minimum distance d.

(c) Consider an orthogonal basis with unit vectors $\mathbf{e_x}$, $\mathbf{e_y}$, and $\mathbf{e_z}$. The initial position vectors of the masses are $\mathbf{r_{01}} = 0$ and $\mathbf{r_{02}} = (2\,\text{m}) \cdot \mathbf{e_x} + (2\,\text{m}) \cdot \mathbf{e_z}$. Their initial velocities are $\mathbf{v_1} = (1\,\text{m/s}) \cdot \mathbf{e_x} + (1\,\text{m/s}) \cdot \mathbf{e_y}$ and $\mathbf{v_2} = (-1\,\text{m/s}) \cdot \mathbf{e_x} + (1\,\text{m/s}) \cdot \mathbf{e_y}$. Calculate t_{min} and d.

Solution. (a) We will work relative to Body 2. The displacement of Body 1 with respect to Body 2 is given by

$$\Delta\mathbf{r} = \mathbf{r_1} - \mathbf{r_2} = (\mathbf{r_{01}} + \mathbf{v_1}t) - (\mathbf{r_{02}} + \mathbf{v_2}t) = (\mathbf{r_{01}} - \mathbf{r_{02}}) + (\mathbf{v_1} - \mathbf{v_2})t \equiv \Delta\mathbf{r_0} + \Delta\mathbf{v}t.$$

Here's what this motion looks like on a diagram at $t = 0$.



The direction of motion for Body 1 is given by the unit vector $\hat{\mathbf{u}} = \Delta\mathbf{v}/|\Delta\mathbf{v}|$. The distance from Body 1 to the point of closest approach is then given by the projection of $\Delta\mathbf{r_0}$ along $\hat{\mathbf{u}}$, i.e. $-\Delta\mathbf{r_0}\cdot\hat{\mathbf{u}}$, where the minus sign corrects for the obtuse angle between the two vectors. This distance is covered in time

$$t_{\min} = -\frac{\Delta\mathbf{r_0}\cdot\hat{\mathbf{u}}}{\Delta\mathbf{v}} = \boxed{-\frac{(\mathbf{r_{01}}-\mathbf{r_{02}})\cdot(\mathbf{v_1}-\mathbf{v_2})}{|\mathbf{v_1}-\mathbf{v_2}|^2}.}$$

(b) The displacement vector $\mathbf{d}$ at the point of closest approach can be formed by adding $(-\Delta\mathbf{r_0}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ to the initial displacement $\Delta\mathbf{r_0}$. Thus

$$d = |\Delta\mathbf{r} - (\Delta\mathbf{r_0}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}| = \boxed{\left|(\mathbf{r_{01}}-\mathbf{r_{02}}) - \frac{(\mathbf{r_{01}}-\mathbf{r_{02}})\cdot(\mathbf{v_1}-\mathbf{v_2})}{|\mathbf{v_1}-\mathbf{v_2}|^2}(\mathbf{v_1}-\mathbf{v_2})\right|.}$$

(c) Finding the scalar products and magnitudes isn't that bad with the values given here. The answers are $\boxed{t_{\min} = 1\,\mathrm{s}}$ and $\boxed{d = 2\,\mathrm{m}.}$

Problem 2. Is the Danube's flow laminar? A river of rectangular cross section has width l and depth h. The river flows laminarly at an angle α to the horizon. Ignore edge effects.

(a) Obtain a formula for the velocity of water $v(z)$ at a distance z from the bottom.

(b) Obtain a formula for the volumetric flow rate of the water Q.

(c) The depth of the Danube is $h = 10\,\mathrm{m}$. The elevation of the water surface is 31 m at Vidin and 16 m at Ruse. The distance to the Danube Delta is 790 km for Vidin and 460 km for Ruse. Find the velocity of water at the surface of the Danube between Vidin and Ruse. Is the Danube's flow laminar?

The acceleration due to gravity is $g = 10\,\mathrm{m/s^2}$, the density of water is $\rho = 1000\,\mathrm{kg/m^3}$ and the viscosity of water is $\eta = 1.0\times10^{-3}\,\mathrm{Pa\,s}$.

Solution. (a) Since the velocity depends only on z, the viscous friction on a layer of area S at some fixed z will be given by $F = \eta S\frac{\mathrm{d}v}{\mathrm{d}z}$. Let us consider the forces on a block of water in contact with the air, whose lower end is at some fixed z. The thickness of the block will be $h-z$, and we'll denote the area of its upper and its lower surface by S. At the lower surface the block experiences a drag force $\eta S\frac{\mathrm{d}v}{\mathrm{d}z}$, but at the upper surface there will be no drag at all. The weight of the block is $\rho g S(h-z)$. The component of the weight along the flow is balanced by the drag, while the component perpendicular to the flow is balanced by a pressure gradient as usual. We conclude that

$$\rho g S(h-z)\sin\alpha = \eta S\frac{\mathrm{d}v}{\mathrm{d}z}.$$

We will integrate this and apply the boundary condition $v(0) = 0$. It is standard to assume that the water at the bottom is static, seeing that the solid surface there offers so much friction that the viscous drag pales in comparison. Then

$$\boxed{v(z) = \frac{\rho g \sin\alpha}{\eta}\left(hz - \frac{z^2}{2}\right).}$$

(b) The volume passing through the cross-section of the river per unit time is

$$Q = \int_0^h v(z) l \, \mathrm{d}z = \boxed{\frac{\rho g l h^3 \sin\alpha}{3\eta}}.$$

(c) If you are familiar with Bulgarian geography, you'll know that the Danube flows approximately in a straight line between Vidin and Ruse. It's reasonable to estimate that

$$\sin\alpha = \frac{31\,\mathrm{m} - 16\,\mathrm{m}}{330\,\mathrm{km}} = 4.5 \times 10^{-5}.$$

The velocity at the surface is then

$$v(h) = \frac{\rho g h^2 \sin\alpha}{2\eta} = \boxed{22\,700\,\mathrm{m/s}.}$$

This is an outlandish value, so the flow cannot be laminar ($\boxed{\text{no}}$).

Problem 3. Pendulum on an inclined plane. A uniform ball of mass m and radius r is tied to a point on an inclined plane using a string of length l. The plane makes an angle α with the horizon. The acceleration due to gravity is g. The ball can only roll without slipping and the torque due to the string's twist can be neglected. Find the period of small oscillations of the ball about its equilibrium position on the inclined plane. The moment of inertia of the ball with respect to an axis passing through its centre of mass is $I_c = \frac{2}{5}mr^2$.

Solution. This is a difficult problem. If you are unfamiliar with 3D rotation, I advise you to read through Chapter 9 of Morin. Problem 9.22 is especially relevant here. I will provide two solutions. The first one uses energies, which is always preferable for problems with complicated oscillations. The second one will use torques.

I. First, we'll need to figure out the direction of the angular velocity vector $\boldsymbol{\omega}$. The neat way to do that is to identify what the instantaneous axis of rotation is by finding two points in the system which are both instantaneously at rest. Because there is no slipping, one such point is the point of contact between the ball and the plane. Another such point is the pivot of the pendulum. The rotation axis has to be the line passing through both of these. At any instant the system performs a pure rotation about this axis with angular velocity ω, its moment of inertia being $\frac{2}{5}mr^2 + mr^2 = \frac{7}{5}mr^2$.

Now we'll introduce $R = \sqrt{(l+r)^2 - r^2}$, which is the distance between the pivot and the point of contact. Let the angle φ describe the rotation of this axis along the plane with respect to the equilibrium position. Since the ball rolls without slipping, we require $\dot{\varphi}R = \omega r$. We'll use this to express the kinetic energy K of the ball in terms of $\dot{\varphi}$. The rotation axis we are working with saves us the hassle of having to account for any translational kinetic energy. It's also a principal axis, so we can just state $K = \frac{1}{2}(\frac{7}{5}mr^2)\omega^2$, which is the same as $K = \frac{1}{2}(\frac{7}{5}mR^2)\dot{\varphi}^2$. Now we will obtain the potential energy of the ball with respect to φ, setting it to zero at equilibrium. When the rotation axis pivots by φ, the ball is raised by $R(1 - \cos\varphi)$ along the plane, so its height increases by $R(1-\cos\varphi)\sin\alpha$. Since φ is small, this corresponds to a potential energy $V = \frac{1}{2}(mgR\sin\alpha)\varphi^2$.

When the total energy is of the form $E = K + V = \frac{1}{2}A\dot{q}^2 + \frac{1}{2}Bq^2$ and is constant, we can differentiate both sides to find that $\ddot{q} + \frac{B}{A}q = 0$, which indicates small oscillations with angular frequency $\sqrt{\frac{B}{A}}$. In our problem we've got $A = \frac{7}{5}mR^2$ and $B = mgR\sin\alpha$, so

$$T = 2\pi\sqrt{\frac{A}{B}} = 2\pi\sqrt{\frac{7R}{5g\sin\alpha}} = \boxed{2\pi\sqrt{\frac{7}{5}\frac{\sqrt{l^2+2lr}}{g\sin\alpha}}}.$$

II. Introduce the orthonormal set $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, such that $\hat{\mathbf{z}}$ points vertically and $\hat{\mathbf{x}}$ is horizontal and along the instantaneous axis of rotation at equilibrium. We'll also denote the unit vector along the instantaneous axis by $\hat{\mathbf{r}}$ (unlike the first three, this one will change with time). We'll also borrow from the notation in the previous solution. The angular momentum of the system is $\mathbf{L} = -\frac{7}{5}mr^2\omega\hat{\mathbf{r}} = -\frac{7}{5}mrR\dot{\varphi}\hat{\mathbf{r}}$. The total torque on the system is then

$$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = -\frac{7}{5}mrR\ddot{\varphi}\hat{\mathbf{r}} - \frac{7}{5}mrR\dot{\varphi}^2\hat{\boldsymbol{\theta}},$$

where $\hat{\boldsymbol{\theta}}$ is a unit vector that lies on the inclined plane and is perpendicular to $\hat{\mathbf{r}}$, such that the time derivative of $\hat{\mathbf{r}}$ is $\dot{\varphi}\hat{\boldsymbol{\theta}}$. Working with respect to the instantenous axis of rotation, the only force that generates a torque is gravity. We want to find the $\hat{\mathbf{r}}$-component of this torque. The lever arm vector is always $r(\cos\alpha\hat{\mathbf{z}}+\sin\alpha\hat{\mathbf{x}})$ and the force is $-mg\hat{\mathbf{z}}$, so the torque is $mgr\sin\alpha\,\hat{\mathbf{y}}$. Its projection along $\hat{\mathbf{r}}$ is $mgr\sin\alpha\sin\varphi$. For small angles this allows us to write the equation

$$-\frac{7}{5}mrR\ddot{\varphi} = mgr\sin\alpha\sin\varphi \quad\Leftrightarrow\quad \ddot{\varphi} + \frac{5g\sin\alpha}{7R}\varphi = 0,$$

which leads us to the same final answer,

$$\boxed{T = 2\pi\sqrt{\frac{7}{5}\frac{\sqrt{l^2+2lr}}{g\sin\alpha}}.}$$

Problem 4. Voltage rectifier. The circuit on Figure 3 is connected to an ideal AC source of RMS voltage $\mathcal{E}_{\text{eff}} = 12\,\text{V}$ and frequency $\nu = 50\,\text{Hz}$. The resistance used is $R = 100\,\Omega$. The diode is ideal, i.e. zero resistance in one direction and infinite resistance in the other.

(a) Sketch the time dependences for the voltage of the source $\mathcal{E}(t)$ and the voltage across the resistor $U(t)$ on the same graph. The graph must include at least one full period of the AC voltage.

(b) Calculate the minimum capacitance $C_{\min}$ for which the voltage fluctuations on the resistor do not exceed $\Delta U = U_{\max} - U_{\min} = 1.0\,\text{V}$.

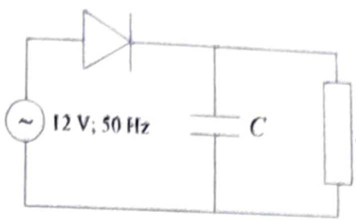


Figure 3

Solution. (a) Let's convert to variables that are actually useful: the amplitude of the source is $\mathcal{E}_0 = \sqrt{2}\mathcal{E}_{\text{eff}}$ and the angular frequency is $\omega = 2\pi\nu$. This is a circuit problem with casework. We should first explore how the circuit behaves in all possible states, and then check how it transitions from one state to another. We'll always set the voltage at the bottom of the AC source, the capacitor, and the resistor to zero. There's no issue with this because only voltage differences matter in Kirchhoff's rules. Consider the case of an open diode. If the voltage of the source is $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t)$, then the potential at the top of the capacitor is also that. The current flowing through the resistor is $\frac{\mathcal{E}(t)}{R}$, and the current through the capacitor is $\frac{\mathrm{d}q}{\mathrm{d}t} = C\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}$. The total current through the diode is then

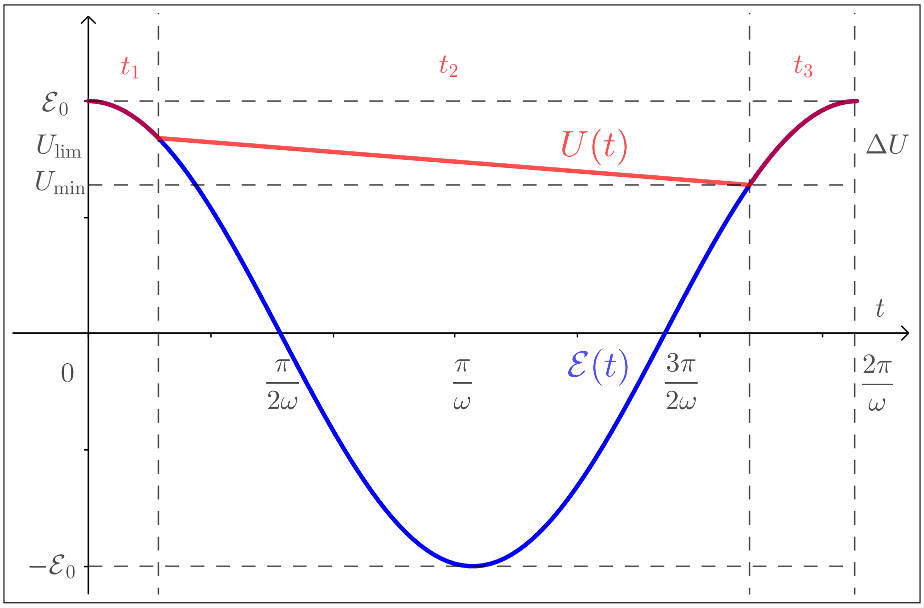
$$I = \mathcal{E}_0 \left(\frac{1}{R} \cos(\omega t) - \omega C \sin(\omega t) \right),$$

which turns negative at time t_1 that satisfies $\tan(\omega t_1) = \frac{1}{\omega RC}$. This corresponds to a potential $U_{\text{lim}} = \mathcal{E}_0 \sqrt{\frac{\omega RC}{\omega RC + 1}}$.

In the case of a closed diode, we have a simple RC circuit. The charge on the upper plate of the capacitor starts off as $q_0 = CU_{\text{lim}}$, and afterwards $\frac{q}{C} - IR = 0$ with $I = -\frac{\mathrm{d}q}{\mathrm{d}t}$, so

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{RC} = 0 \quad \Rightarrow \quad q(t) = q_0 e^{-t/RC} \quad \Rightarrow \quad U(t) = U_{\text{lim}} e^{-t/RC}.$$

This exponential decrease shouldn't bring the potential down to less than $U_{\text{max}} - \Delta U$. The maximum voltage here is of course $\mathcal{E}_0$ at $t = 0$. The decrease will continue until $U(t)$ once again becomes equal to $\mathcal{E}(t)$. After that the diode reopens. Here's a plot:



The exponential decay is shown as a straight line because we have a small decrease in $U(t)$, whereby $U_{\text{lim}} e^{-t/RC} \approx U_{\text{lim}}(1 - \frac{t}{RC})$.

(b) Let's find the times t_1, t_2, and t_3. In the limiting case t_3 obeys $\mathcal{E}_0 \cos(\omega t_3) = \mathcal{E}_0 - \Delta U$. The numerical values of $\mathcal{E}_0$ and ΔU imply $\omega t_3 \ll 1$ (and likewise for t_1), so $t_3 = \frac{1}{\omega}\sqrt{\frac{2\Delta U}{\mathcal{E}_0}}$. We previously found $\tan(\omega t_1) = \frac{1}{\omega RC}$, but $\omega t_1 \ll 1$, so $\omega t_1 = \frac{1}{\omega RC}$. Since $U_{\text{lim}} = \mathcal{E}_0 \cos(\omega t_1)$, we can approximate $U_{\text{lim}} = \mathcal{E}_0(1 - \frac{(\omega t_1)^2}{2})$. The minimum voltage in a cycle is $U_{\text{lim}}(1 - \frac{t_2}{RC})$, which gives us

$$\mathcal{E}_0 \left(1 - \frac{(\omega t_1)^2}{2}\right)\left(1 - \frac{t_2}{RC}\right) = \mathcal{E}_0 - \Delta U.$$

Now we can substitute $t_2 = \frac{2\pi}{\omega} - t_1 - t_3$ and neglect the highest order term ($\sim t_1^2 t_2$). Then we set $u = \frac{1}{\omega RC}$ to get the following equation for u:

$$\frac{1}{2}u^2 - \left(2\pi - \sqrt{\frac{2\Delta U}{\mathcal{E}_0}}\right)u + \left(\frac{\Delta U}{\mathcal{E}_0}\right) = 0.$$

The roots are

$$u = \left(2\pi - \sqrt{\frac{2\Delta U}{\mathcal{E}_0}}\right) \pm \sqrt{\left(2\pi - \sqrt{\frac{2\Delta U}{\mathcal{E}_0}}\right)^2 - \left(\frac{2\Delta U}{\mathcal{E}_0}\right)}.$$

When $\Delta U = 0$, the slope of the exponential decay should go to zero, i.e. we should have $\frac{1}{RC} \to 0$, meaning $u \to 0$. Thus the physical root is the lesser one of the two. We can now apply the binomial approximation, and our result reduces to

$$u = \frac{\Delta U/\mathcal{E}_0}{2\pi - \sqrt{\frac{2\Delta U}{\mathcal{E}_0}}}.$$

The denominator is approximately 2π, and we end up with

$$\boxed{C_{\min} = \frac{1}{\nu R}\left(\frac{\sqrt{2}\mathcal{E}_{\text{eff}}}{\Delta U}\right) = 3.4\,\text{mF}.}$$

For a similar problem, see NBPhO 2008-6.

Problem 5. Convoluted grating. A diffraction grating has alternating slits of different length (wide, thin, wide…). The distance between adjacent slits is a, as shown on Figure 4. Monochromatic light of wavelength λ ($\lambda \ll a$) is normally incident on the grating. We observe the diffraction pattern at a large distance L ($L \gg a$) from the grating. If the wide slits are closed and the thin slits are left open, the maxima are of essentially equal intensity I_0. If we close the thin slits and leave the wide ones open, the maxima have intensity $2I_0$.

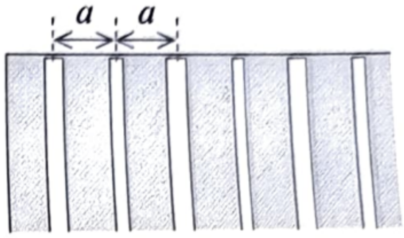


Figure 4

Now we leave all the slits open. Find the distance between the maxima Δx and the intensity of the maxima I_k in terms of their order k.

Solution. Solving this from first principles is tough, but we can use some general knowledge about diffraction gratings. We'll work with phasors. Take the case where only the thin slits are open. Let's assume that each thin slit corresponds to a phasor of amplitude A_1. This means that you get one phasor of amplitude A_1 per length $2a$ along the grating. Whenever there is a maximum of the diffraction pattern, all phasors are aligned, and the observed intensity is $kA_1^2 = I_0$, where k is some constant which depends on the grating's parameters. For example, if there are N slits in total, $k \propto N^2$, but this doesn't really matter.

Likewise, if only the wide slits are open, you get a phasor of amplitude A_2 per length $2a$ along the grating. Maxima occur where the phasors align, but this time $kA_2^2 = 2I_0$. Note that k is the same because the geometry of the two setups is indistinguishable.

Now consider the case where both types of slits are open. For a segment of length $2a$ along the grating, this time you get one phasor of amplitude A_1 and one phasor of amplitude A_2. The angle between them is $\phi = \frac{2\pi}{\lambda}a\sin\theta$, where θ is the angle between the normal of the slits and the direction towards the point on the screen that you're looking at. To get a maximum from a diffraction grating ($N \to \infty$), all phasors must be aligned, or else the addition of the phasors turns chaotic. There are two distinct ways for alignment to happen. One is when $\phi = (2m+1)\pi$ for integer m, and the other is when $\phi = (2m)\pi$ for integer m. In the first case every pair of phasors A_1 and A_2 sums to $A_2 - A_1$, but these pairs can collectively be treated like the phasors in the single-slit cases. This means that

$$I_{2m+1} = k(A_2 - A_1)^2 = k\left(\sqrt{\frac{2I_0}{k}} - \sqrt{\frac{I_0}{k}}\right)^2 = (3 - 2\sqrt{2})I_0 \approx \boxed{0.17I_0.}$$

Similarly, in the second case each pair sums to $A_2 + A_1$, so

$$I_{2m} = k(A_2 + A_1)^2 = k\left(\sqrt{\frac{2I_0}{k}} + \sqrt{\frac{I_0}{k}}\right)^2 = (3 + 2\sqrt{2})I_0 \approx \boxed{5.83I_0.}$$

The distance between the maxima on the screen corresponds to a phase increment of $\Delta\phi = \pi$. This is the equivalent to a change in $\sin\theta$ of magnitude $\Delta(\sin\theta) = \frac{\lambda}{2a}$. The coordinates of the maxima on the screen are given by $x = L\tan\theta \approx L\sin\theta$, from which we find

$$\boxed{\Delta x = \frac{\lambda L}{2a}.}$$

Problem 6. Optical fiber. A point source is placed at one end of an optical fiber, as shown on Figure 5. It emits a short light pulse of energy $E_0 = 5\,\mu J$, radiated isotropically within the fiber (uniformly in all directions inside the fiber). The fiber is $L = 10.0\,m$ long and its refractive index is $n = 1.50$. The fiber is surrounded by air of refractive index 1.

(a) Find the energy E_1 and the length Δt of the light pulse that reaches the other end of the fiber.

(b) Find an expression for the instantaneous power $P(t)$ of the light pulse at the other end of the fiber. The time t is measured starting from the emission of the pulse.

Figure 5

Solution. (a) Consider a ray which makes an angle θ with the normal of the optical fiber's surface. There are two things that can happen. If $\sin\theta < \frac{1}{n}$, the ray gets refracted at the surface of the fiber and leaves the system. But when $\sin\theta \geq \sin\theta_{crit} = \frac{1}{n}$, the ray experiences total internal reflection, and it will continue to propagate along the fiber, always at an angle of incidence θ. Then, the rays that reach the other end of the fiber will correspond to $\theta \in [\theta_{crit}; \pi/2]$. At the point source, these are all contained within a spherical cap centered at the axis of the fiber and spreading out to an angle $\alpha = \frac{\pi}{2} - \theta_{crit}$. The solid angle of this cap is

$$\Omega = 2\pi(1 - \cos\alpha) = 2\pi(1 - \sin\theta_{crit}) = 2\pi\left(1 - \frac{1}{n}\right).$$

Now we use $\frac{\Omega}{2\pi} = \frac{E_1}{E_0}$, which gives us $\boxed{E_1 = \frac{n-1}{n}E_0 = 1.67\,\mu J.}$

As for the duration of the pulse, we should note that a ray with an angle of incidence θ will have covered a total distance $\frac{l}{\sin\theta}$ after propagating by l along the axis of the fiber. This implies that the first and last rays to arrive will respectively require time

$$t_1 = \frac{L}{c/n} \quad \text{and} \quad t_2 = \left(\frac{1}{\sin\theta_{\text{crit}}}\right)\frac{L}{c/n}.$$

The duration is then $\Delta t = t_2 - t_1 = \boxed{\frac{n(n-1)L}{c} = 2.5 \times 10^{-8}\,\text{s}.}$

(b) Let us first state that $P(t) = 0$ for $t < t_1$ and $t > t_2$. Now onto the problem of finding $P(t)$ for $t \in [t_1;\, t_2]$. The time of arrival for the rays is given by $t = \frac{1}{\sin\theta}\frac{nL}{c}$. The difference in arrival times $\mathrm{d}t$ depends on the difference in incidence angles $\mathrm{d}\theta$ as follows:

$$\mathrm{d}t = -\frac{\cos\theta}{\sin^2\theta}\left(\frac{nL}{c}\right)\mathrm{d}\theta.$$

We will later use this in determining

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}\Omega}\cdot\frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{E_0}{2\pi}\cdot\frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Finding $\frac{\mathrm{d}\Omega}{\mathrm{d}\theta}$ is again a matter of working with spherical caps. We have already seen that $\Omega = 2\pi(1-\sin\theta)$, so $\frac{\mathrm{d}\Omega}{\mathrm{d}\theta} = -2\pi\cos\theta$. Then

$$P = \frac{E_0}{2\pi}\cdot(-2\pi\cos\theta)\cdot\left(-\frac{\sin^2\theta}{\cos\theta}\frac{c}{nL}\right) = \frac{E_0 c}{nL}\left(\frac{1}{\sin\theta}\right)^2 = \boxed{\left(\frac{nLE_0}{c}\right)t^{-2}.}$$

A nice way to check our answer is to calculate $\int_{nL/c}^{n^2L/c} P\,\mathrm{d}t$. This gives us E_1, as indeed it should.

Problem 7. Relativistic dynamics.

(a) Express the relativistic momentum p of a particle of mass m in terms of its kinetic energy K.

Consider two particles of rest mass m. The first particle has kinetic energy K_0. It collides elastically with the second particle, which is at rest, and is deflected through an angle θ.

(b) Find the kinetic energy K_1 of the second particle after the collision in terms of K_0 and θ.

Solution. (a) We will set $c = 1$ in our calculations and restore the c's at the end. Let the total energy of the particle be E. The kinetic energy is $K = E - m$. Hence

$$p^2 + m^2 = E^2 = (K+m)^2 \quad\Rightarrow\quad p = \sqrt{K(K+2m)} \quad\Leftrightarrow\quad \boxed{p = \frac{1}{c}\sqrt{K(K+2mc^2)}.}$$

(b) We will follow what the problem suggests and work with kinetic energies rather than total energies. The rest masses are constant in an elastic collision, so we can write $K_0 = K_1 + K_2$. Momentum is also conserved, so $\mathbf{p_0} = \mathbf{p_1} + \mathbf{p_2}$. We now have to involve θ, the angle between $\mathbf{p_0}$ and $\mathbf{p_1}$. The easiest way to do this is by squaring $\mathbf{p_2} = \mathbf{p_0} - \mathbf{p_1}$ to get $p_2^2 = p_0^2 + p_1^2 - 2p_0p_1\cos\theta$. Then we end up with

$$(K_0 - K_1)(K_0 - K_1 + 2m) = K_0(K_0+2m) + K_1(K_1+2m) - 2\sqrt{K_0K_1(K_0+2m)(K_1+2m)}\cos\theta.$$

In relativistic dynamics problems most things cancel out after squaring, so this isn't as horrible as it looks. Indeed, it simplifies to

$$(K_0 + 2m)K_1 = \sqrt{K_0 K_1 (K_0 + 2m)(K_1 + 2m)} \cos\theta.$$

After squaring this and rearranging, we obtain

$$K_1 = \frac{2mK_0 \cos^2\theta}{2m + K_0 \sin^2\theta} \quad \Leftrightarrow \quad \boxed{K_1 = \frac{K_0 \cos^2\theta}{1 + \frac{K_0}{2mc^2}\sin^2\theta}.}$$

You are encouraged to check that this result is valid in the special cases $\theta = 0$ and $\theta = \pi/2$.

Problem 8. Predictions of the Bohr model.

(a) Calculate the magnetic field $\mathbf{B}$ (direction and magnitude) at the centre of a hydrogen atom due to an electron on the first Bohr orbit.

(b) Since the proton possesses a magnetic moment

$$\mu_p = \frac{e\hbar}{2m_p}$$

with possible projections $\pm\mu_p$ along the direction of the field, the ground state energy of the atom will change. Estimate the splitting of the the energy level in that case.

Solution. (a) Let us denote by $\hat{\mathbf{n}}$ the unit vector directed along the electron's angular velocity $\boldsymbol{\omega}$. The electron orbits with a period $T = 2\pi/\omega$, which gives rise to a current $I = \frac{-e}{T}$. If the radius of its orbit is a, this current generates a magnetic field similarly to a circular current loop,

$$\mathbf{B} = \frac{\mu_0 I}{2a}\hat{\mathbf{n}} = -\frac{\mu_0 e}{2aT}\hat{\mathbf{n}} = -\frac{\mu_0 \omega e}{4\pi a}\hat{\mathbf{n}}.$$

We still need to find ω and a. The electrostatic force acts as a centripetal force, so $m_e\omega^2 a = \left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{a}\right)$. Apart from that, the orbital angular momentum for the n-th Bohr orbit is $n\hbar$, and in our case $m_e\omega a^2 = \hbar$. This set of equations yields

$$a = \frac{4\pi\varepsilon_0}{m_e}\left(\frac{\hbar}{e}\right)^2, \qquad \omega = \left(\frac{m_e}{\hbar^3}\right)\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2,$$

for a final result of

$$\mathbf{B} = -\left(\frac{\mu_0 e}{4\pi}\right)\left(\frac{m_e^2}{\hbar^5}\right)\left(\frac{e^2}{4\pi\varepsilon}\right)^3 \hat{\mathbf{n}} = \boxed{-(12.5\,\mathrm{T})\hat{\mathbf{n}}.}$$

(b) A magnetic moment $\mathbf{p}$ in a magnetic field $\mathbf{B}$ is associated with a potential energy $U = -\mathbf{p}\cdot\mathbf{B}$. The total energy then gets shifted by $\pm\mu_p B$, which corresponds to a total difference of

$$\Delta E = 2\mu_p B = \left(\frac{\mu_0 e^2}{4\pi}\right)\left(\frac{m_e^2}{m_p\hbar^4}\right)\left(\frac{e^2}{4\pi\varepsilon}\right)^3 = 1.27\times 10^{-25}\,\mathrm{J} = \boxed{0.79\,\mu\mathrm{eV}.}$$

Problem 9. Neutron cooling. A fast neutron experiences elastic central collisons in a medium that acts as a moderator. Find the number of collisions necessary for a neutron of energy 1 MeV to reach the thermal velocity for temperature $T = 300\,\mathrm{K}$ in graphite.

Solution. The value $E_0 = 1\,\text{MeV}$ is negligible compared to the neutron's rest energy $m_n c^2 = 938\,\text{MeV}$, so this is a classical problem about an electron with initial kinetic energy E_0. The neutron is subject to a series of central elastic collisions with static graphite atoms of mass $M = 12m_n$. Each collision will result in a decrease in the neutron's velocity. If the speed before a collision is v, then the speed after the collision is $v' = \frac{M-m}{M+m}v$. This is a standard result, for which a derivation has been presented in the solution of Short Exam 1.

The kinetic energy scales as v^2, so after n collisions it has gone down to $E = E_0\left(\frac{M-m}{M+m}\right)^{2n}$. If the neutron is at thermal velocity, its kinetic energy is $E_{\text{th}} = \frac{3}{2}k_B T = 0.039\,\text{eV}$. We need to find the smallest n for which $E < E_{\text{th}}$. Then

$$E_0\left(\frac{11}{13}\right)^{2n} < \frac{3}{2}k_B T, \qquad (2n)\ln\left(\frac{11}{13}\right) < \ln\left(\frac{3k_B T}{2E_0}\right).$$

We're looking for an integer, so

$$\boxed{n_{\min} = \left\lceil \frac{1}{2}\frac{\ln\left(\frac{3k_BT}{2E_0}\right)}{\ln\left(\frac{11}{13}\right)} \right\rceil = 52.}$$

Problem 10. Heat engine. A heat engine operates on a reversible cycle consisting of several processes. In the process 1-2 the molar heat capacity is proportional to the temperature and increases from $C_1 = 20\,\text{J}\,\text{K}^{-1}\,\text{mol}^{-1}$ to $C_2 = 50\,\text{J}\,\text{K}^{-1}\,\text{mol}^{-1}$. The next process 2-3 is adiabatic. The last process 3-1 is isothermal. Find the efficiency of the cycle. Note that the equation of state of the working substance is unknown.

Solution. We're not allowed to work with a specific equation of state (e.g. that of an ideal gas), but we can still get an answer using entropy arguments, and it will be valid for any gas. The cycle is reversible, meaning that the total entropy change in a complete cycle is zero. We will now track the entropy changes at each stage.

Working with n moles of gas, for 1-2 we're given that $C = \frac{1}{n}\frac{\mathrm{d}Q}{\mathrm{d}T} = aT$ for some constant a. The initial and final temperatures of the gas T_1 and T_2 can be found through $aT_1 = C_1$ and $aT_2 = C_2$, which gives us $\frac{T_2}{T_1} = \frac{C_2}{C_1}$. Since the entropy change in a reversible process is defined as $\mathrm{d}S = \mathrm{d}Q/T$, we can write $\mathrm{d}S = na\,\mathrm{d}T$, for a total of $\Delta S_{12} = na(T_2 - T_1) = n(C_2 - C_1)$.

The adiabatic process involves no heat transfer, and hence no entropy change, $\Delta S_{23} = 0$. Finally, the isothermal process is executed at a constant temperature T_1, so the entropy change is $\Delta S_{31} = \int \frac{\mathrm{d}Q}{T_1} = \frac{Q_{31}}{T_1}$. We are left with the following relation:

$$\Delta S_{12} + \Delta S_{23} + \Delta S_{31} = 0 \quad \Rightarrow \quad Q_{31} = -n(C_2 - C_1)T_1.$$

Let's also find the heat transfer in the first process using $\frac{1}{n}\frac{\mathrm{d}Q}{\mathrm{d}T} = aT$. We get

$$Q_{12} = na\left(\frac{T_2^2}{2} - \frac{T_1^2}{2}\right) = \frac{n}{2}(C_2T_2 - C_1T_1) = \frac{n}{2}\left(\frac{C_2^2 - C_1^2}{C_1}\right)T_1.$$

The efficiency of the cycle η can be found from the heat input Q_{in} and the waste heat Q_{out}:

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}.$$

Looking at the signs of Q_{12} and Q_{31} we identify $Q_{\text{in}} \Leftrightarrow Q_{12}$ and $Q_{\text{out}} \Leftrightarrow |Q_{31}|$, which gives us

$$\eta = 1 - \frac{n(C_2 - C_1)}{\frac{n}{2}\left(\frac{C_2^2 - C_1^2}{C_1}\right) T_1} = 1 - \frac{2C_1}{C_1 + C_2} = \boxed{\frac{C_2 - C_1}{C_2 + C_1} = 0.43.}$$

Constants:

Boltzmann constant	k_B	1.38×10^{-23} J/K
Gas constant	R	$8.31\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_A	$6.02 \times 10^{23}\,\mathrm{mol^{-1}}$
Elementary charge	e	1.60×10^{-19} C
Vacuum permeability	μ_0	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Speed of light in vacuum	c	3.00×10^{8} m/s
Electron mass	m_e	9.11×10^{-31} kg
Proton mass	m_p	1.67×10^{-27} kg
Neutron mass	m_n	1.67×10^{-27} kg
Reduced Planck constant	$\hbar$	1.05×10^{-34} J s

Experimental Exam

Problem 1. All about gravity.

Equipment: Golf ball (mass 46 g, diameter 43 mm), table tennis ball (mass 2.7 g, diameter 40 mm), stopwatch, tape measure, tape measure, three-legged stool, wooden blocks, ruler, graph paper

Task 1. Measuring the acceleration due to gravity.
Using the wooden blocks, tilt the table at different angles to the horizon. By measuring the rolling time of the golf ball on the table, find the acceleration due to gravity g. Assume the golf ball is homogeneous. **(6.0 pt)**

Task 2. Measuring the coefficient of restitution for inelastic collisions.
Study the bouncing of the table tennis ball on the three-legged stool. Measure the dependence of total bouncing time of the ball on its initial height. Assume that the ball has stopped when you can no longer hear the sound from the collisions. Using your data, calculate the restitution coefficient of the partially inelastic collisions between the ball and the stool. **(9.0 pt)**

Relevant theory:
1) When a homogeneous ball rolls without slipping on an inclined surface that makes an angle α with the horizon, the time taken for the ball to move a distance l starting from rest is given by

$$t = \sqrt{\frac{14l}{5g\sin\alpha}}.$$

2) The restitution coefficient for a collision between two bodies is defined as $v_{\text{rel, after}}/v_{\text{rel, before}}$, where $v_{\text{rel, before}}$ and $v_{\text{rel, after}}$ are the relative velocities before and after the collision. Assume that this coefficient does not depend on the relative velocity of the bodies.

3) If a body is left to bounce from an initial height h, its total bouncing time (from the instant it is dropped until the instant it comes to rest) is given by

$$T = \frac{1+k}{1-k}\sqrt{\frac{2h}{g}}.$$

Constants and formulae:

Acceleration due to gravity	$g = 9.81\,\text{m/s}^2$
Reynolds number	$\text{Re} = \frac{\rho v L}{\eta}$
Critical Reynolds number for fluid flow around a ball	$\text{Re}_{\text{cr}} \approx 1$
Stokes' law	$F_{\text{drag}} = 6\pi\eta r v$
Newton's law	$F_{\text{drag}} = \frac{1}{2}C\rho S v^2$
Drag coefficient for a sphere	$C \approx \frac{1}{2}$
Density of air at normal temperature and pressure	$\rho = 1.29\,\text{kg/m}^3$
Viscosity of air at normal temperature and pressure	$\eta = 1.86 \times 10^{-5}\,\text{Pa s}$