Short Exam 2

Problem. A solid dielectric[1] sphere of radius R and relative permittivity $\varepsilon_r = 1$ is charged uniformly, such that the total charge is Q. The sphere is surrounded by a dielectric medium, also of permittivity $\varepsilon_r = 1$. The charge density in the medium varies with distance as $\rho(r) = \alpha/r$, where α is a known constant (and naturally, $r > R$).

(a) Find the charge Q which would make the electric field have a constant magnitude across the medium.

(b) Find the magnitude of this electric field E.

(c) A point charge q of mass m is located at a distance L from the centre of the sphere (q and Q are like). The charge is launched with a velocity v at an angle $\beta \in (0, \frac{\pi}{2})$ from the direction towards the centre of the sphere. For what velocities v will the charge avoid hitting the sphere? Neglect any frictional forces.

Solution. (a) The word 'dielectric' seems scary, but in this case it's just used to indicate a medium that holds the charge in place. The electric susceptibility of a material is given by $\chi_e = \varepsilon_r - 1$. In our case this is zero, so the polarisation $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ is also zero. Ordinarily the polarisation gives rise to a surface bound charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ and a volume bound charge density $\rho_b = -\nabla \cdot \mathbf{P}$, but these are again zero, so we only need to consider the embedded charges from the problem statement.

Consider a sphere of radius r. The total charge within is

$$q_{\text{in}} = Q + \int_R^r \left(\frac{\alpha}{r}\right) 4\pi r^2 \mathrm{d}r = Q + 2\pi\alpha(r^2 - R^2).$$

The electric field $\mathbf{E} = E\hat{\mathbf{r}}$ is identical everywhere on the surface of this sphere, so we can find it using Gauss's law:

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q + 2\pi\alpha(r^2 - R^2)}{4\pi\varepsilon_0 r^2}.$$

We now require E to be constant. To eliminate the r-dependence, we need $\boxed{Q = 2\pi\alpha R^2}$.

(b) After substituting the appropriate Q, we obtain $\boxed{E = \frac{\alpha}{2\varepsilon_0}}$. Or, in vector form, $\mathbf{E} = \frac{\alpha}{2\varepsilon_0}\hat{\mathbf{r}}$.

(c) The electric field outside the sphere is associated with a potential $\varphi = -\int \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\alpha r}{2\varepsilon_0}$, where we chose $\varphi = 0$ at $r = 0$. The total energy of the charge is then

$$\mathcal{E} = \frac{mu^2}{2} - \frac{q\alpha r}{2\varepsilon_0} = \frac{mv^2}{2} - \frac{q\alpha L}{2\varepsilon_0}.$$

We note that q and $Q = 2\pi\alpha R^2$ are like, so the potential energy term is always negative no matter the sign of α. Thus, the velocity of the charge would decrease closer to the sphere. Next, because the electric field is radial, the angular momentum of the charge is also conserved:

$$L = mu_\tau r = mvL \sin\beta.$$

[1] In the original problem statement, the sphere was conducting. In that case, Part (c) becomes a complicated image charge problem which reduces to a fifth order equation with no analytical solution. Surely this isn't what the problem author intended.

We will use this to find an expression for the distance of closest approach r_{min}. At that point the velocity of the charge is purely tangential, so we can substitute $u = \frac{vL\sin\beta}{r_{\text{min}}}$ in the energy conservation statement. We find

$$\frac{mv^2}{2}\left(\left(\frac{L\sin\beta}{r_{\text{min}}}\right)^2 - 1\right) = \frac{q\alpha}{2\varepsilon_0}(L - r_{\text{min}}).$$

For the charge to avoid the sphere, we want $r_{\text{min}} > R$. Since the electrostatic force is repulsive, we'd prefer a small velocity v so that there's enough time for the charge to be sent away from the sphere. Then, for certain angles β there's a limiting value for v above which the charge will strike the sphere. We can find it by setting $r_{\text{min}} = R$:

$$v^2 = \frac{q\alpha}{m\varepsilon_0}\frac{(L-R)R^2}{(L\sin\beta)^2 - R^2}.$$

We see that for $\sin\beta > R/L$, this v is undefined. This makes sense, because for those β the charge isn't headed towards the sphere to begin with, so there's no chance of hitting it whatever v is. We can now assemble the following answer for the values of v at which we avoid the sphere:

$$\boxed{v < \sqrt{\frac{q\alpha}{m\varepsilon_0}\frac{(L-R)R^2}{(L\sin\beta)^2 - R^2}}}, \qquad \sin\beta < R/L,$$

$$\boxed{\text{Any } v}, \qquad \sin\beta > R/L.$$

Theoretical Exam

Problem 4. A resistance $R = 2\,\Omega$ and a nonlinear lightbulb are connected in parallel. They are connected to a battery of EMF $E = 4\,\mathrm{V}$ and internal resistance $r = 0.5\,\Omega$. The I-V curve of the lightbulb is given in the table. Find the power dissipated at the lightbulb.

U, V	I, A
0.50	0.60
1.00	1.00
1.50	1.30
2.00	1.55
2.50	1.75
3.00	1.90

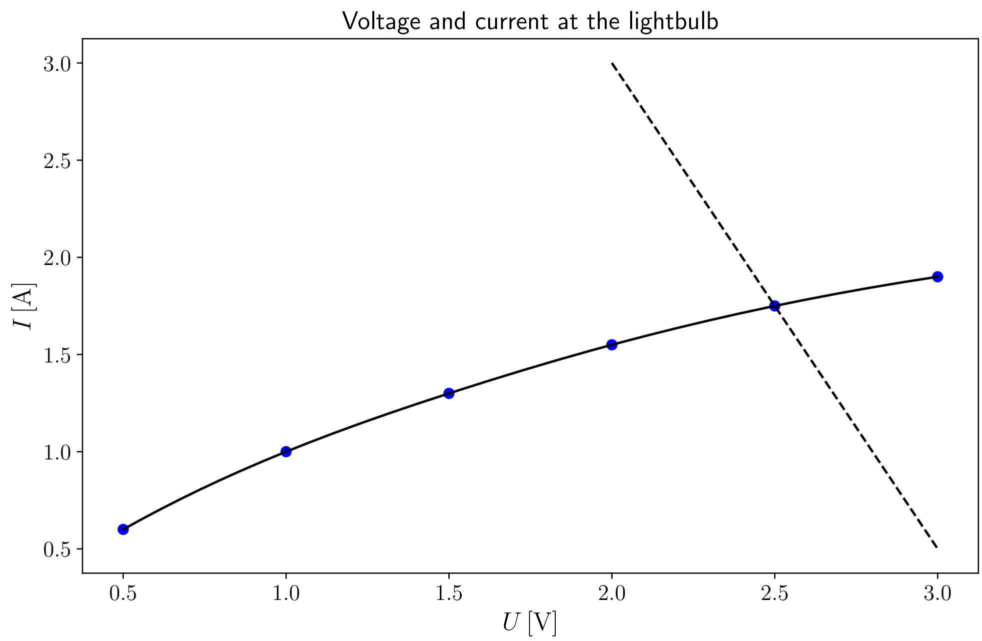
Solution. Using Kirchoff's rules, we will find how the voltage across the lightbulb U is related to the current I passing through it. The resistance R is connected in parallel to the lightbulb, so the voltage across it is also equal to U, and the current through it is $\frac{U}{R}$. The total current through the battery is therefore $I_{\mathrm{tot}} = I + \frac{U}{R}$. Then, the loop equation for the circuit is

$$E - I_{\mathrm{tot}} r - U = 0,$$

which is the same as

$$I = \frac{E}{r} - \frac{R+r}{Rr} U = (8\,\mathrm{A}) - (2.5\,\Omega^{-1})\,U. \tag{1}$$

On the other hand, I and U must also follow the I-V curve of the lightbulb. The state of the lightbulb will then correspond to the point on the I-V curve which lies on (1):



We find that the voltage is $U = 2.50\,\mathrm{V}$, while the current is $I = 1.75\,\mathrm{A}$. The dissipated power is $P = UI = \boxed{4.38\,\mathrm{W}}$.

Problem 5. The distance between a light source and a screen is L. There are two positions where a lens can be put between the source and the screen so that a clear image of the source is projected on the screen. The ratio of these images' sizes is k ($k > 1$). Find a formula for the focal length of the lens F in terms of L and k. Calculate F for $L = 180\,\mathrm{cm}$ and $k = 4$.

Solution. In this problem there is a real image on the screen, meaning that we don't need to consider the case of a lens placed behind the source. Denote the distance between the source and the lens by x. The distance from the lens to the screen is then $L - x$, and the thin lens formula gives us

$$\frac{1}{x} + \frac{1}{L-x} = \frac{1}{F} \quad \Rightarrow \quad x^2 - Lx + LF = 0.$$

The solutions to this quadratic equation are as follows:

$$x_{1,2} = \frac{L \pm \sqrt{L^2 - 4LF}}{2}.$$

These are valid provided that $L > 4F$; else we can't have an image on the screen due to this lens. We also note that the two solutions sum to L, so they are like a pair of x and $L - x$. After tracking the rays that pass through the centre of the lens, we see that in each case the magnification of the object is $M = \frac{L-x}{x}$. For the larger solution x_1, we have $M_1 < 1$, while for the other solution we have $M_2 > 1$. The sizes of the two images are then related by

$$k = \frac{M_2}{M_1} = \frac{L - x_2}{x_2}\frac{x_1}{L - x_1} = \left(\frac{x_1}{x_2}\right)^2.$$

Now our goal is to find f from the equation

$$\sqrt{k} = \frac{L + \sqrt{L^2 - 4LF}}{L - \sqrt{L^2 - 4LF}}.$$

We reach $\sqrt{L^2 - 4LF} = \frac{\sqrt{k}-1}{\sqrt{k}+1}L$. We square this to get

$$\boxed{F = \frac{\sqrt{k}}{(\sqrt{k}+1)^2}L.}$$

This is less than $L/4$, as expected. For $L = 180\,\text{cm}$ and $k = 4$ our formula yields $\boxed{F = 20\,\text{cm}}$.

Problem 6. Two parallel beams of light are normally incident on a diffraction grating of period $d = 8\,\mu\text{m}$. Their wavelengths are λ_1 and λ_2, both in the range ($400\,\text{nm} - 760\,\text{nm}$). We observe their diffraction patterns for all angles $\theta \in (-90°, 90°)$. The patterns' maxima coincide at an angle θ_0 with $\sin\theta_0 = 0.2725$. One of the patterns has 8 maxima more than the other. Find λ_1 and λ_2. How many possible solutions are there?

Solution. We will take $\lambda_2 > \lambda_1$ without loss of generality. The angles of the maxima for light of wavelength λ_i can be found from $d\sin\theta = k\lambda_i$, $k \in \mathbb{Z}$. For the maximum at θ_0 we have $d\sin\theta_0 = 2180\,\text{nm} = k_1\lambda_1 = k_2\lambda_2$. The integers that result in wavelengths within the allowed range are $k = 3, 4, 5$. They correspond to $\lambda = 726.6\,\text{nm}, 545\,\text{nm}, 436\,\text{nm}$.

The number of maxima at each wavelength is $N = 2n_i + 1$, where n_i is the largest integer for which the equation $d\sin\theta = n_i\lambda_i$ admits a solution for θ. We'd need $n_i\lambda_i < d$ for that, so $n_i = \left\lfloor \frac{d}{\lambda_i} \right\rfloor$. The $2n_i + 1$ here accounts for the central maximum and the maxima on both sides. We're told that $N_1 - N_2 = 8$, which reduces to

$$\left\lfloor \frac{d}{\lambda_1} \right\rfloor - \left\lfloor \frac{d}{\lambda_2} \right\rfloor = 4.$$

Substituting the available wavelengths λ_i, the terms on the left can be either 11, 14 or 18. The only pair that works is 18 and 14, and this corresponds to $\boxed{\lambda_1 = 436\,\text{nm}}$ and $\boxed{\lambda_2 = 545\,\text{nm}}$. This is the $\boxed{\text{only solution}}$.

Experimental Exam

Problem 1. Deceiving the compass.

Equipment:

1. Compass with degree markings (Figure 1).
2. Small neodymium magnet in the shape of a rectangular cuboid. The edges of the magnet are, in ascending order,

$$a = 5\,\text{mm};\ b = 10\,\text{mm};\ c = 15\,\text{mm}.$$

   The mass of the magnet is 5.6 g, distributed uniformly. Its magnetisation is parallel to one of the edges, meaning that the poles are on the surfaces perpendicular to that edge.
3. Plastic ruler.
4. Stopwatch.
5. String.
6. Graph paper.
7. Marker pen.

Figure 1: Fingers.

The aim of this problem is to determine the horizontal component B_0 of the Earth's magnetic field, as well as the dipole moment p of the neodymium magnet.

Tasks:

(a) Design and describe an experiment that can be used to find whether the magnets are magnetised along a, b, or c. Also find the surfaces corresponding to the north pole (N) and the south pole (S) of the magnet. Use the marker to indicate them on the magnet using the letters N and S.

(b) Study the deviation of the compass from the north-south axis at different distances r between the needle and the neodymium magnet. The distance is measured between the centres of the objects. Using the data, plot a graph in the appropriate variables and use it to find the ratio p/B_0. Estimate your error.

(c) Design and carry out an experiment from which you can determine the product pB_0. Estimate your error.

(d) Calculate the horizontal component of the Earth's magnetic field B_0 and the dipole moment p of the magnet. Estimate your errors.

Note: Do not falsify your data so as to obtain the textbook value of B_0. You will be in for a nasty surprise!

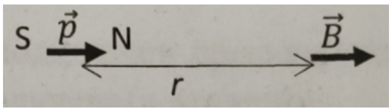
Relevant theory:
1) A magnetic dipole moment $\mathbf{p}$ can be ascribed to any current I circulating on a closed loop of surface S:

$$\mathbf{p} = IS\mathbf{n},$$

where $\mathbf{n}$ is a unit vector normal to the surface S. The units for dipole moment are $[\mathrm{A\,m^2}]$. The atoms in permanent magnets can be treated as tiny current loops, meaning that permanent magnents also possess a net dipole moment. It points from their south pole to their north pole.

2) The point dipole approximation holds when the distance to a dipole r is much larger than its characteristic length. Then the field along the axis of a dipole is given by

$$\mathbf{B} = \frac{\mu_0 \mathbf{p}}{2\pi r^3}.$$



3) When a dipole is placed in an external magnetic field $\mathbf{B}_{\text{ext}}$, it experiences a torque

$$\mathbf{M} = \mathbf{p} \times \mathbf{B}_{\text{ext}} \qquad \text{(i.e. of magnitude } M = pB_{\text{ext}} \sin\theta).$$

that tends to rotate it towards the direction of the field.

4) The moments of inertia of a uniform rectangular cuboid about axes passing through its centre parallel to the edges a, b, and c, are respectively given by

$$I_a = \frac{1}{12}m(b^2 + c^2); \quad I_b = \frac{1}{12}m(c^2 + a^2); \quad I_c = \frac{1}{12}m(a^2 + b^2).$$

Problem 2. Conductivity of liquids.

Equipment:
Alternating voltage source, 2 multimeters, wires, paperclips of unknown radius r, tub, ruler, 3 bottles with 1.5 l of tap water in each, kitchen scale, table salt (NaCl), tape, scissors, aluminium foil (be careful with the sharp edges), funnel, graph paper.

All inductances and capacitances in the circuits below are negligible. Record all measurements in tables. Write down your results in the answer sheet.

Task 1. Conductivity of tap water.
Consider two identical infinite cylindrical conductors of radius r. The distance between the conductors is L. If the conductors pierce an infinite weakly conducting layer normally, some resistance R will be measured between them. If the thickness of the layer is h and its conductivity is σ, the resistance is given by

$$R = \frac{C}{\sigma h} \ln\left(\frac{L}{r}\right), \tag{2}$$

where C is an unknown constant.

Pour the three bottles of water into the tub. Assemble a circuit that will allow you to measure the resistance between two vertical paperclips in a layer of water. The multimeters should be connected in a way that allows you to measure small resistances. Use AC voltage of RMS value $U = 4\,\mathrm{V}$ and frequency $\nu = 2\,\mathrm{kHz}$.

1. Measure the resistance between the paperclips R at different distances $L \in [1\,\mathrm{cm}, 5\,\mathrm{cm}]$ in steps of $0.5\,\mathrm{cm}$. Keep the paperclips as far away as possible from the sides of the tub. Present your results in a table. **(4.0 pt)**

2. Stick a sheet of foil to each of the two smaller sides of the tub. Measure the resistance of the tap water between the sheets R. Find the conductivity of the tap water σ. Record the necessary length measurements in the answer sheet as well. **(2.0 pt)**

3. Using Equation (1) and your value for the conductivity of tap water σ from Task 1(b), plot your data from Task 1(a) in appropriate variables and find the radius of the paperclips r, as well as the constant C. **(3.0 pt)**

Task 2. Conductivity of salt salutions.
Add salt to the water and study four solutions of mass fractions $c = 0.50\,\%$, $1.00\,\%$, $1.50\,\%$, $2.00\,\%$ one after the other.

1. Use the sheets of foil from Task 1(b) to measure the conductivity σ of these solutions. Assume that it is given by

$$\sigma = \alpha c. \tag{3}$$

   Plot a graph of the conductivity σ against the mass fraction c of the salt solutions. Using the graph, find the parameter α. **(4.0 pt)**

2. Equation (2) does not agree well with the experimental data. Instead assume a more general dependence of the form

$$\sigma = \beta c^n. \tag{4}$$

   Plot another graph in the appropriate variables and use it to find the number n. **(2.0 pt)**

Call the examiner if you suspect that a multimeter's fuse has blown, or in case of any other technical difficulties.