Short Exam 1

Problem. The planets E and M are in circular orbits around the star S. Their orbital radii are respectively $r_E = 150 \times 10^6$ km and $r_M = 230 \times 10^6$ km. The rotational period of planet E around the star S is $T_E = 365$ d.

(a) Find the rotational period T_M of planet M around the star S, in days.

The inhabitants of planet E wish to get to planet M. The spaceship is to travel with the engines turned off on an orbit tangent to both the orbits of planet E and planet M. The gravitational forces between the spaceship and the planets can be neglected.

(b) Find the duration T_{EM} of the flight from planet E to planet M, in days.

(c) Find the angle $\angle ESM$ at the instant when the spaceship takes off from planet E.

(d) Find the time T_2 after which the angle between the planets (i.e. their relative position) is again suitable for launching an identical spaceship from planet E to planet M.

(e) Find the velocity v of the spaceship at launch with respect to the star (in km/s).

Solution. (a) By Kepler's third law,

$$\frac{r_E^3}{T_E^2} = \frac{r_M^3}{T_M^2} \quad \Rightarrow \quad \boxed{T_M = T_E \left(\frac{r_M}{r_E}\right)^{3/2} = 693\,\text{d.}}$$

(b) On this elliptical orbit, the minimum distance to the Sun is r_E, while the maximum distance is r_M. Then, the semi-major axis is $a = \frac{r_E + r_M}{2}$, and the orbital period T can be found from

$$\frac{a^3}{T^2} = \frac{r_E^3}{T_E^2}.$$

The flight corresponds to half of the ellipse and takes time $\frac{1}{2}T$, so

$$\boxed{T_{EM} = \frac{1}{2} T_E \left(\frac{r_E + r_M}{2r_E}\right)^{3/2} = 260\,\text{d.}}$$

(c) The spaceship's destination is a point M' for which $\angle ESM' = 180°$. The spaceship should be launched in such a way that planet M arrives at M' exactly at the end of the flight. At the launch of the spaceship M should have an angle of $\frac{T_{EM}}{T_M} \cdot 360° = 135°$ to cover before reaching M'. In conclusion,

$$\angle ESM = 180° - \frac{T_{EM}}{T_M} \cdot 360° = \boxed{180° \left(1 - \left(\frac{r_E + r_M}{2r_M}\right)^{3/2}\right) = 45°.}$$

(d) The relative position of the planets is the same when the segment SE rotates by 360° with respect to the segment SM. The former's angular velocity (in degrees per unit time) is $\frac{360°}{T_E}$, while the latter's angular velocity is $\frac{360°}{T_M}$. The relative angular velocity is simply their difference. Then, T_2 satisfies

$$\left(\frac{360°}{T_E} - \frac{360°}{T_M}\right) T_2 = 360°.$$

It follows that

$$T_2 = \frac{T_M T_E}{T_M - T_E} = \boxed{T_E \left(\frac{r_M^{3/2}}{r_M^{3/2} - r_E^{3/2}} \right) = 771\,\text{d}.}$$

(e) Let the velocity of the spaceship at M' be v'. If the star's mass is m, the conservation of energy statement for points E and M' is

$$\frac{v^2}{2} - \frac{Gm}{r_E} = \frac{v'^2}{2} - \frac{Gm}{r_M}.$$

Additionally, we have conservation of angular momentum:

$$v r_E = v' r_M.$$

Solving the system of equations, we find

$$v = \sqrt{\frac{2Gm}{r_E + r_M} \frac{r_M}{r_E}}.$$

Kepler's third law gives us $\frac{r_E^3}{T_E^2} = \frac{Gm}{4\pi^2}$, and then

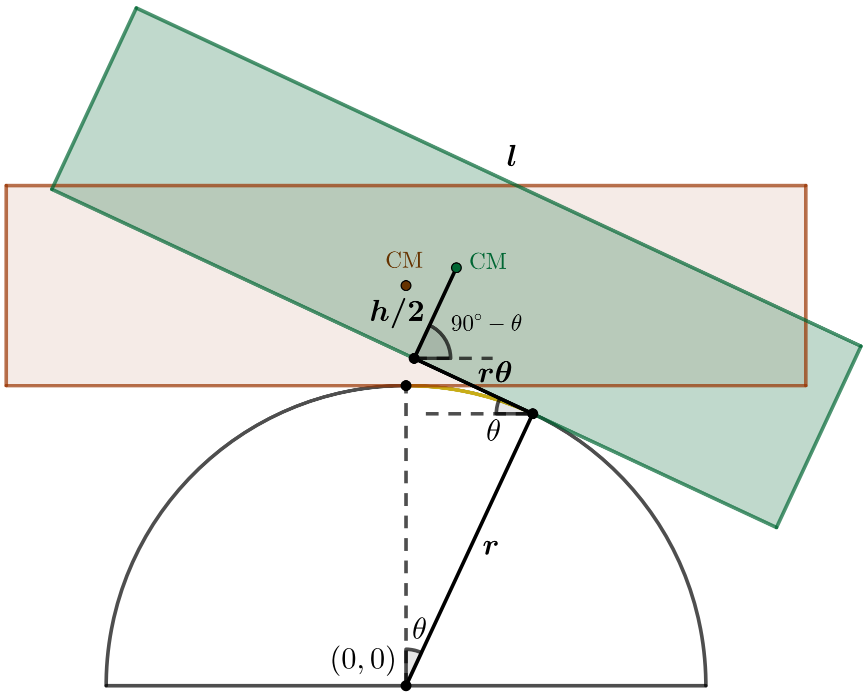
$$\boxed{v = \frac{2\pi r_E}{T_E} \sqrt{\frac{2r_M}{r_E + r_M}} = 33\,\text{km/s}.}$$

Problem 1. A half-cylinder of radius r lies on the ground with its flat surface down. A uniform rod of rectangular cross section is placed symmetrically on top of the half-cylinder perpendicularly to its axis. The rod has mass m, length l, and height h. The acceleration due to gravity is g.

(a) How should the parameters be related if the rod's equilibrium is stable?

(b) If the equilibrium is stable, find a formula for the oscillation period of the rod T when it is displaced from its equilibrium position.The rod does not slip on the half-cylinder's surface.

Solution. (a) This is Problem 8.4 ('Leaning rectangle') from Morin's textbook. I will provide a solution of my own here. Let's place the coordinate origin $(x, y) = (0, 0)$ at the centre of the half-cylinder. When the rod is in equilibrium, the centre of mass (CM) of the rod is at $(0, r + \frac{h}{2})$. Now let's rotate the rod along the cylinder so that it ends up at a small angle θ to the horizon. As this happens, the contact point between the two objects sweeps an arc $r\theta$ on the surface of the cylinder. Since there is no slipping, the objects' surfaces are essentially interlocked, so the total shift in the contact point's position is equal on both surfaces. Thus, on the rod, the new contact point is shifted by exactly $r\theta$ with respect to the middle:



The position of the CM can then be described with

$$x = \left(r + \frac{h}{2}\right)\sin\theta - (r\theta)\cos\theta,$$
$$y = \left(r + \frac{h}{2}\right)\cos\theta + (r\theta)\sin\theta.$$

The shift of the CM's height, accurate to second order in θ, is

$$\Delta y = y - \left(r + \frac{h}{2}\right) \approx \left(r + \frac{h}{2}\right)\left(1 - \frac{\theta^2}{2}\right) + r\theta^2 - \left(r + \frac{h}{2}\right) = \frac{1}{2}\left(r - \frac{h}{2}\right)\theta^2,$$

and hence the potential energy, taking it as zero in equilibrium, is $E_p = mg\Delta y = \frac{1}{2}mg\left(r - \frac{h}{2}\right)\theta^2$. For the equilibrium of the rod to be stable, neighbouring configurations should have a higher potential energy, meaning that the term in front of the θ^2 has to be positive. The condition for stable equilibrium is then $\boxed{r > h/2}$.

(b) The velocity of the centre of mass can be found by differentiating the expressions for its position:

$$v_x = \left(\left(r + \frac{h}{2}\right)\cos\theta - r\cos\theta + r\theta\sin\theta\right)\dot{\theta},$$
$$v_y = \left(-\left(r + \frac{h}{2}\right)\sin\theta + r\sin\theta + r\theta\cos\theta\right)\dot{\theta}.$$

After applying the small-angle approximations, the only remaining large term is $v_x = \frac{h}{2}\dot{\theta}$. The kinetic energy of the rod can be broken up into a translational term $\frac{mv_{\text{CM}}^2}{2}$ and a rotational term $\frac{I\dot{\theta}^2}{2}$, where the moment of inertia of the rod about its centre of mass is $I = \frac{1}{12}m(l^2 + h^2)$, as derived in TST Problem 2012-2. In total, kinetic energy is

$$E_k = \frac{1}{2}m\left(\frac{h^2}{4} + \frac{l^2}{12} + \frac{h^2}{12}\right)\dot{\theta}^2 = \frac{1}{2}m\left(\frac{h^2}{3} + \frac{l^2}{12}\right)\dot{\theta}^2.$$

Next, given that total energy (which has the form $A\dot{\theta}^2 + B\theta^2$) is a conserved quantity, we can differentiate it to find the equation of motion of the rod:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_k + E_p\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(A\dot{\theta}^2 + B\theta^2\right) = 2\dot{\theta}\left(A\ddot{\theta} + B\theta\right) = 0.$$

Setting the stuff in the parentheses to zero, we get simple harmonic oscillations with an angular frequency $\omega = \sqrt{\frac{B}{A}}$. In this problem we have $A = \frac{1}{2}m\left(\frac{h^2}{3} + \frac{l^2}{12}\right)$ and $B = \frac{1}{2}mg\left(r - \frac{h}{2}\right)$, so the period is

$$T = \frac{2\pi}{\omega} = \boxed{2\pi\sqrt{\frac{4h^2 + l^2}{6g(2r - h)}}.}$$

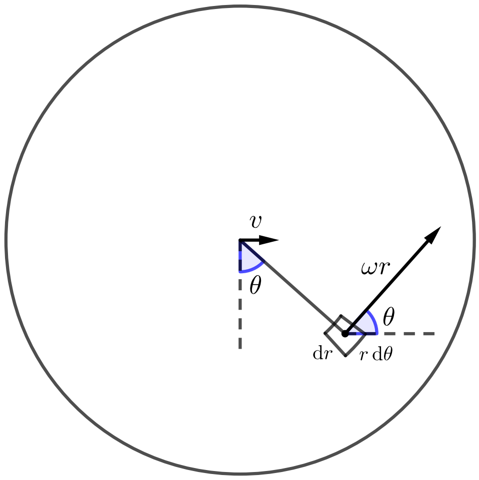
Problem 2. A disc of radius R and mass m is placed on a horizontal surface. Initially the disc rotates with angular velocity ω about its axis of symmetry. The initial velocity of its centre of mass is v (where $\omega R \gg v$). Find the initial friction force F acting on the disc. The coefficient of friction between the disc and the surface is k. The acceleration due to gravity is g.

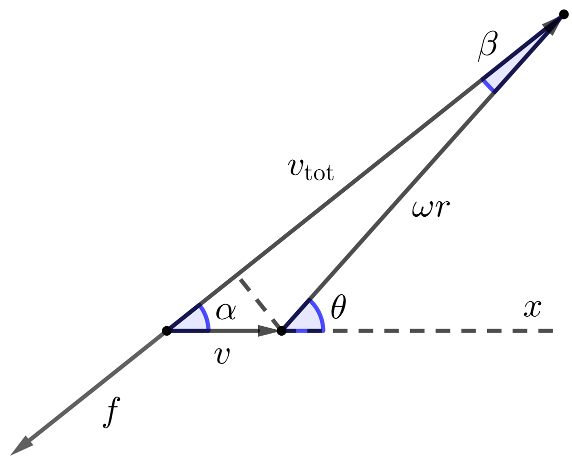
Solution. We'll examine the friction on a piece of the disc at polar coordinates (r, θ), as shown. Its total velocity is a sum of the centre-of-mass velocity $\mathbf{v}$ and the rotational velocity $\boldsymbol{\omega} \times \mathbf{r}$. Given that $\mathbf{v}$ acts like a small correction to $\boldsymbol{\omega} \times \mathbf{r}$, the total velocity (and hence the friction force) will make some angle α with the x-axis that will equal θ plus some little adjustment β. We see that $\beta = \frac{v\sin\theta}{\omega r} \ll 1$ and $\alpha = \theta - \beta$. Then, it holds that

$$\cos\alpha = \cos(\theta - \beta) \approx \cos\theta + \beta\sin\theta,$$
$$\sin\alpha = \sin(\theta - \beta) \approx \sin\theta - \beta\cos\theta.$$

The mass per unit area for the disc is $\sigma = \frac{m}{\pi R^2}$, from where the friction force on a small element can be expressed by $\mathrm{d}f = k\sigma(r\mathrm{d}r\mathrm{d}\theta)g$. The components of this force are

$$\mathrm{d}f_x = -k\sigma(r\mathrm{d}r\mathrm{d}\theta)g\cos\alpha = -k\sigma g(r\mathrm{d}r\mathrm{d}\theta)\left(\cos\theta + \frac{v}{\omega r}\sin^2\theta\right),$$
$$\mathrm{d}f_y = -k\sigma(r\mathrm{d}r\mathrm{d}\theta)g\sin\alpha = -k\sigma g(r\mathrm{d}r\mathrm{d}\theta)\left(\sin\theta - \frac{v}{\omega r}\sin\theta\cos\theta\right).$$





To get the total force, we integrate these from 0 to 2π in θ and between 0 and R in r. The only angular term that doesn't integrate to zero is the $\sin^2\theta$ in $\mathrm{d}f_x$. It's well-known that the average value of $\sin^2\theta$ in any block of length π is $\frac{1}{2}$, so the integral between 0 and 2π evaluates to $\frac{1}{2}\cdot 2\pi = \pi$. We reach

$$f = -\int_0^R k\sigma g\left(\frac{v}{\omega}\right)\pi\mathrm{d}r = \boxed{-kmg\left(\frac{v}{\omega R}\right)}.$$

This is a small quantity and it reduces to zero when $v \to 0$, as expected.

Problem 3. Two point masses, $m = 1\,\mathrm{kg}$ each, lie on a smooth horizontal surface. The masses are connected by a stiff massless spring with relaxed length $d = 1\,\mathrm{m}$ and spring constant $k = 1\,\mathrm{N/m}$. Initially the spring is relaxed, one of the masses is at rest, and the other is given a horizontal velocity $v = 1\,\mathrm{m/s}$ perpendicular to the spring. Find the maximum elongation of the spring x, accurate to $1\,\mathrm{mm}$.

Solution. The surface is frictionless, so there are no external horizontal forces. Thus, following the collision, the velocity of the centre of mass (CM) will remain constant. This velocity is equal to $\frac{mv}{m+m} = \frac{v}{2}$. Let's work in the reference frame in which the CM is at rest. This frame is inertial, and the initial velocities of the masses there are tangential, opposite, and equal to $\frac{v}{2}$ each. The spring will now start stretching, and the masses will also acquire some radial velocities v_r.

The motion that ensues is subject to a few conservation laws. Firstly, conservation of momentum implies that the velocities of the masses are always directed opposite to each other. Next, the pull from the spring is radial, so the angular momentum with respect to the CM is conserved as well. If at some instant the tangential velocities are v_τ each, while the extension of the spring is x', we have

$$2mv_\tau\left(\frac{d+x'}{2}\right) = 2m\left(\frac{v}{2}\right)\left(\frac{d}{2}\right).$$

Finally, the energy is also constant:

$$2\left(\frac{m(v_r^2+v_\tau^2)}{2}\right) + \frac{kx'^2}{2} = 2\left(\frac{m(v/2)^2}{2}\right).$$

When the spring is stretched the most, the balls have zero radial velocity, and the conservation laws take the form

$$v_\tau(d+x) = \frac{v}{2}d, \qquad v_\tau^2 + \frac{kx^2}{2m} = \left(\frac{v}{2}\right)^2,$$

from which we conclude that x obeys

$$\frac{kx^2}{2m} = \left(\frac{v}{2}\right)^2 \left(1 - \left(\frac{d}{d+x}\right)^2\right).$$

This cannot be solved analytically, and we're just looking for the numerical value of x, so we plug in all the numbers to reduce this to an equation for x in metres:

$$2x^2 = 1 - \left(\frac{1}{1+x}\right)^2.$$

We should be accurate to 0.001 in x, so there's a long road ahead of us. The only solution is $x = 0.537\,\mathrm{m}$ (more precisely, $0.536\,97\,\mathrm{m}$). We state this in mm: $\boxed{x = 537\,\mathrm{mm}}$.

Experimental Exam

Problem 1. Bifilar torsional pendulum.

Equipment:
2 rulers (each of unknown mass m), 2 coins (each of mass $M = 7.00\,\mathrm{g}$), tape measure, stopwatch, string, scissors, tape.

A bifilar torsional pendulum consists of a homogeneous rod of mass m and length L attached to two strings of length h at points equidistant from the centre of mass. The distance between the strings is d. The pendulum oscillates with period T about a vertical axis passing through its centre of mass. The moment of inertia I of a rod of mass m and length l about an axis passing through its centre of mass perpendicularly to the rod is $I = \frac{1}{12}ml^2$. Record your results in the answer sheet.

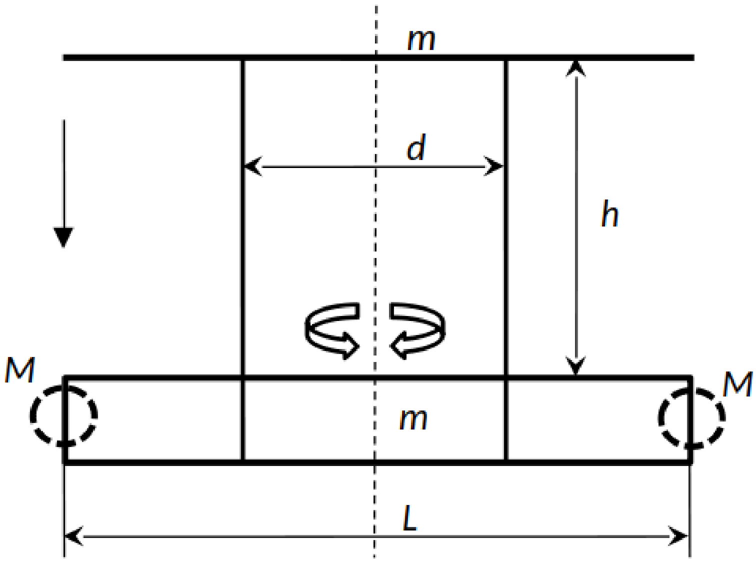


Figure 1

(a) Find the centre of mass of the ruler. Write down the division of the ruler where it is located. What is the value of L that you will be working with? **(0.5 pt)**

(b) Suspend the ruler as described above (see Figure 1, the plane of the ruler should be vertical). If the period T depends on the length of the strings h as $T \propto h^n$, find the number n experimentally. Round your value to one of the numbers $\left(\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\right)$. **(3.5 pt)**

(c) If the period T depends on the distance between the strings d as $T \propto d^k$, find the number k experimentally. Round your value to one of the numbers $\left(\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\right)$. **(3.5 pt)**

(d) All in all, the period of the pendulum is given by

$$T = 2\pi C \frac{L}{\sqrt{g}} h^n d^k,$$

where C is some number. Find the value of C experimentally. **(3.5 pt)**

(e) Let the period of the pendulum be $T(0)$ for some constant d and h. If we attach a coin to each end of the ruler (so that the centre of each coin is exactly on the rim of the ruler), the period becomes $T(M)$. The period of the pendulum $T(\mu)$ depends on its mass μ and its moment of inertia $I(\mu)$ as $T(\mu) \propto \sqrt{\frac{I(\mu)}{\mu}}$. Find a formula $m = f(M, T(0), T(M))$ from which the mass of the ruler m can be determined. Take the necessary measurements and find m. **(4.0 pt)**