Short Exam 2

Problem. Efficiency of a circuit. Consider the circuit on Figure 1. The voltage source, the ammeters, and the voltmeter are ideal. The resistances R_1 and R_2 are constant, while the resistance R_x may vary between zero and infinity. The efficiency of the circuit η is defined as the ratio of the power dissipated at R_x to the total power dissipated in the circuit.

(a) Find an expression for η in terms of R_1, R_2, and R_x. **(2.5 pt)**

(b) Find the value of R_x which maximizes η, in terms of R_1 and R_2. **(1.5 pt)**

(c) Find an expression for the maximum η in terms of the ratio $k = R_2/R_1$. **(0.5 pt)**

(d) Find the value of η (in percent) for $k = 1$. **(0.5 pt)**

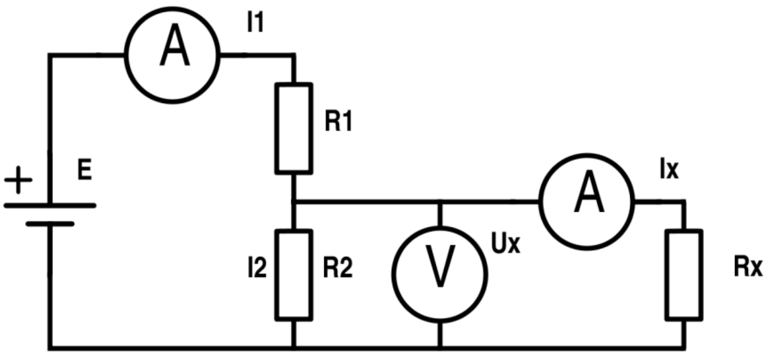

Figure 1

Solution. (a) Firstly, let's note that there will be no power dissipation at the ammeters and voltmeters because their resistances are zero and infinity, respectively. Now, using the notation on the diagram, the power at R_x is U_xI_x, while the power for the whole circuit equals the power supplied by the battery, which is EI_1. Then we have $\eta = \frac{U_xI_x}{EI_1}$, and all that is left is to find how the currents and voltages are related. The current I_1 will split into I_x and $I_2 = I_1 - I_x$, such that $I_xR_x = I_2R_2$. It follows that $I_x = \frac{I_1R_2}{R_x+R_2}$, which brings us to $\eta = \frac{R_2}{R_x+R_2}\frac{U_x}{E}$. The equivalent resistance of R_2 and R_x is $R_{\text{eq}} = \frac{R_2R_x}{R_2+R_x}$. The voltage drop at R_{eq} is $U_x = \frac{ER_{\text{eq}}}{R_1+R_{\text{eq}}}$, and thus

$$\eta = \frac{R_2}{R_x+R_2}\cdot\frac{R_{\text{eq}}}{R_1+R_{\text{eq}}} = \boxed{\frac{R_2^2R_x}{(R_2+R_x)(R_1R_2+R_1R_x+R_2R_x)}}.$$

(b) The efficiency η is maximised when its inverse $1/\eta$ is minimised. Equivalently, we want to maximise

$$\left(\frac{R_2}{R_x}+1\right)\left(R_x(R_1+R_2)+R_1R_2\right).$$

The part of this expression that depends on R_x is

$$\left(\frac{R_2}{R_x}\right)R_1R_2 + R_x(R_1+R_2).$$

Now we can either take the derivative or use the AM-GM inequality to find that the expression is minimised when the two terms are equal, which happens at

$$\boxed{R_x = \sqrt{\frac{R_1}{R_1+R_2}}R_2.}$$

(c) We substitute $R_1 = \frac{R_2}{k}$ and $R_x = \frac{R_2}{\sqrt{1+k}}$ into the expression for η. After some tedious manipulations, we get

$$\boxed{\eta = \frac{k}{2\sqrt{1+k} + (2+k)}.}$$

(d) The answer is

$$\eta = \frac{1}{3+2\sqrt{2}} = \boxed{17.2\,\%.}$$

If you have time, it's a good idea to solve the special case $k = 1$ separately and confirm that the numerical answer is in agreement with the general formula.

Theoretical Exam

Problem 4. Alloy density. Gold-copper alloys Au_xCu_{1-x} (of number fraction x) have a face-centred cubic lattice, as shown on Figure 2. The atoms of copper and gold are randomly distributed on the lattice points. The density of pure gold is $\rho_{Au} = 19.30\,\mathrm{g/cm^3}$, its lattice constant is $a_{Au} = 4.078\,\text{Å}$, while the density of pure copper is $\rho_{Cu} = 8.96\,\mathrm{g/cm^3}$ and its lattice constant is $a_{Cu} = 3.615\,\text{Å}$. Assume that the lattice constant is proportional to x. Let y be the mass fraction of the alloys Au_xCu_{1-x}, such that $y = \frac{m_{Au}}{m_{Cu}+m_{Au}}$.

1. Find the number fraction in terms of the mass fraction, $x = f(y)$. **(1.0 pt)**
2. Find the density in terms of the mass fraction, $\rho_{Au_xCu_{1-x}} = f(y)$. **(1.0 pt)**
3. Find the density of a gold-copper alloy for mass fractions $y = 0.5$ and $y = 0.8$. **(1.0 pt)**

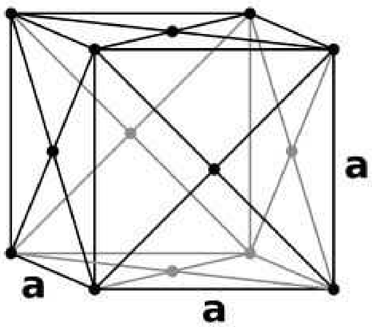

Figure 2

Solution. (a) Each unit cell is a cube of side a. Within the cell, there are 8 atoms at the vertices which are shared between 8 cubes each, and there are 6 atoms at the faces which are shared between 2 cubes each. In total there are $8 \cdot \frac{1}{8} + 6 \cdot \frac{1}{2} = 4$ atoms per unit cell. This means that $\rho_{Au} a_{Au}^3 = 4M_{Au}$ and $\rho_{Cu} a_{Cu}^3 = 4M_{Cu}$, where M_{Au} and M_{Cu} are the masses of a single gold or copper atom.

The mass fraction can be found by comparing the total masses of each element:

$$y = \frac{m_{Au}}{m_{Cu} + m_{Au}} = \frac{M_{Au}x}{M_{Au}x + M_{Cu}(1-x)}.$$

We rearrange this to get x in terms of y:

$$x = \frac{M_{Cu}y}{M_{Cu}y + M_{Au}(1-y)} = \frac{y}{y + \frac{M_{Au}}{M_{Cu}}(1-y)} = \boxed{\frac{y}{y + \left(\frac{\rho_{Au}a_{Au}^3}{\rho_{Cu}a_{Cu}^3}\right)(1-y)}.}$$

(b) The lattice constant a is proportional to x, and we know that $a = a_{\mathrm{Cu}}$ at $x = 0$ (no gold) and $a = a_{\mathrm{Au}}$ at $x = 1$ (no copper). The formula for a in terms of x should then be

$$a = a_{\mathrm{Cu}} + (a_{\mathrm{Au}} - a_{\mathrm{Cu}})x.$$

The density can be found as the mass of the atoms in a cell, divided by the volume of a cell. The total mass of the atoms is equal to that of just the gold atoms, divided by y:

$$\rho = \frac{M_{\mathrm{cell}}}{a^3} = \frac{M_{\mathrm{cell}}(\mathrm{Au})}{a^3}\left(\frac{1}{y}\right).$$

As discussed above, 4 sites with gold atoms would have a total mass of $\rho_{\mathrm{Au}} a_{\mathrm{Au}}^3$. However, only a fraction x of the sites are actually occupied by gold atoms, so

$$\rho = \frac{\rho_{\mathrm{Au}} a_{\mathrm{Au}}^3}{a^3}\left(\frac{x}{y}\right) = \frac{\rho_{\mathrm{Au}} a_{\mathrm{Au}}^3}{(a_{\mathrm{Cu}} + (a_{\mathrm{Au}} - a_{\mathrm{Cu}})x)^3}\left(\frac{x}{y}\right) = \rho_{\mathrm{Au}}\left(\frac{a_{\mathrm{Cu}}}{a_{\mathrm{Au}}} + \left(1 - \frac{a_{\mathrm{Cu}}}{a_{\mathrm{Au}}}\right)x\right)^{-3}\left(\frac{x}{y}\right).$$

We're still not done, because the expression should be in terms of y only. Using our result in (a), we can reach

$$\boxed{\rho = \rho_{\mathrm{Au}}\left(y + \left(\frac{\rho_{\mathrm{Au}} a_{\mathrm{Au}}^3}{\rho_{\mathrm{Cu}} a_{\mathrm{Cu}}^3}\right)(1-y)\right)^{-1}\left(\frac{a_{\mathrm{Cu}}}{a_{\mathrm{Au}}} + \left(1 - \frac{a_{\mathrm{Cu}}}{a_{\mathrm{Au}}}\right)\left(1 + \frac{\rho_{\mathrm{Au}} a_{\mathrm{Au}}^3}{\rho_{\mathrm{Cu}} a_{\mathrm{Cu}}^3}\cdot\frac{1-y}{y}\right)^{-1}\right)^{-3}.}$$

This is awful, and it is essential to do a safety check. The expression should reduce to ρ_{Cu} for $y = 0$ and ρ_{Au} for $y = 1$. And indeed, it does.

(c) To prepare for the calculation, you should first write down $\frac{a_{\mathrm{Cu}}}{a_{\mathrm{Au}}} = 0.887$ and $\frac{\rho_{\mathrm{Au}} a_{\mathrm{Au}}^3}{\rho_{\mathrm{Cu}} a_{\mathrm{Cu}}^3} = 3.092$. It is safer to compute this in chunks rather than in one go. Commit the intermediate values to paper in case you slip up with the calculator. The answers for $y = 0.5$ and $y = 0.8$ are $\boxed{\rho = 12.34\,\mathrm{g/cm^3}}$ and $\boxed{\rho = 15.84\,\mathrm{g/cm^3}}$, respectively.

Problem 5. A microscopic model for resistance. The simplest microscopic model for resistance assumes that the free electrons in metals are accelerated from rest by the external electric field for time τ, after which they collide with the ionic lattice. This τ is called the mean free time. After colliding with the lattice, the electrons lose all of their velocity. Then they begin accelerating again.

(a) The electron number density in a metal is n and its resistivity is ρ. Express τ in terms of these parameters. **(1.0 pt)**

(b) Find the ratio $P/\frac{\Delta E_{\mathrm{kin}}}{\Delta t}$ between the power P dissipated in a conductor of length l and cross section S, and the kinetic energy $\frac{\Delta E_{\mathrm{kin}}}{\Delta t}$ lost by the electrons (as heat) per unit time. **(1.0 pt)**

(c) Find the mean free time for the electrons in aluminium. Aluminium ions have charge $q = +3e$. The atomic mass of aluminium is $A = 27.0$, its density is $\mu = 2.70\,\mathrm{g/cm^3}$, and its resistivity is $\rho = 28.2 \times 10^{-9}\,\Omega\,\mathrm{m}$. **(1.0 pt)**

Solution. (a) The time-averaged current in a conductor of cross section S due to electrons of mean velocity is given by $\langle v\rangle$ is $\langle I\rangle = neS\langle v\rangle$, which can be seen by tracking the number of charge carriers passing through an area S. The time-averaged current density will then be $\langle j\rangle = ne\langle v\rangle$. Let's denote the electric field in the conductor by E. Now, each free electron moves with a uniform acceleration $a = \frac{eE}{m_e}$ for time τ until it collides with the lattice. This means that it covers a distance $s = \frac{a\tau^2}{2}$ in time τ, which corresponds to a mean velocity of $\langle v \rangle = \frac{s}{\tau} = \frac{a\tau}{2}$. The current density is then

$$\langle j \rangle = \frac{nea\tau}{2} = \frac{ne^2 E\tau}{2m_e}.$$

Conversely, Ohm's law gives us $\langle j \rangle = \frac{E}{\rho}$. We match the two expressions to find

$$\boxed{\tau = \frac{2m_e}{ne^2\rho}.}$$

(b) The standard way to calculate the power is

$$P = I^2 R = I^2 \left(\frac{\rho l}{S}\right) = \langle j \rangle^2 \rho (Sl).$$

Meanwhile, on a small scale, each electron loses energy $\frac{m_e (a\tau)^2}{2}$ per time τ, so the total energy loss rate in a conductor of volume $V = Sl$ is

$$\frac{\Delta E_{\text{kin}}}{\Delta t} = n(Sl)\frac{m_e a^2 \tau}{2}.$$

The ratio we are looking for is

$$P \Big/ \left(\frac{\Delta E_{\text{kin}}}{\Delta t}\right) = \frac{2\rho \langle j \rangle^2}{nm_e a^2 \tau} = \frac{2\rho E^2}{\rho^2 n m_e \left(\frac{e^2 E^2}{m_e^2}\right)\left(\frac{2m_e}{ne^2\rho}\right)} = \boxed{1.}$$

(c) The mass of a single aluminium atom is Au, so the number density of the atoms is $\frac{\mu}{Au}$. There are three free electrons per atom, so the electron number density is $n = \frac{3\mu}{Au}$. Then

$$\boxed{\tau = \frac{2Aum_e}{3\mu e^2 \rho} = 1.4 \times 10^{-14}\,\text{s}.}$$

Problem 6. Charged disc in a magnetic field. A uniform dielectric disc of mass m and charge q is initially at rest. The disc is placed in a magnetic field parallel to its axis. The field is arbitrary, $B = B(t)$, with $B(0) = 0$. Find the time dependence of the angular velocity of the disc $\omega(t)$.

Solution. Denote the radius of the disc by R and its charge density by $\sigma = \frac{q}{\pi R^2}$. In keeping with the symmetry of the problem, consider a circular loop of radius $r < R$. This loop is pierced by a magnetic flux $\Phi = B\pi r^2$. The magnetic field is always changing, so by Faraday's law we have an electric field E along the loop:

$$E(r) \cdot 2\pi r = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}B}{\mathrm{d}t}\pi r^2.$$

A surface element at r which carries a charge $\mathrm{d}q$ will then feel a force $E\,\mathrm{d}q$, which creates a torque $Er\,\mathrm{d}q$ about the rotation axis. The net torque from the ring between r and $r + \mathrm{d}r$ is

$$\mathrm{d}M = Er(2\pi r\sigma\,\mathrm{d}r) = -\frac{\mathrm{d}B}{\mathrm{d}t}\left(\frac{\pi r^2}{2\pi r}\right) r\,(2\pi r\sigma\,\mathrm{d}r).$$

The total torque from the whole disc is

$$M = \int \mathrm{d}M = -\frac{\mathrm{d}B}{\mathrm{d}t}\sigma\left(\frac{\pi R^4}{4}\right) = -\frac{\mathrm{d}B}{\mathrm{d}t}\left(\frac{qR^2}{4}\right).$$

This is equal to $I\frac{\mathrm{d}\omega}{\mathrm{d}t}$, where $I = \frac{1}{2}mR^2$ is the moment of inertia of the disc. The radius cancels out, and we're left with

$$\mathrm{d}\omega = -\mathrm{d}B\frac{q}{2m}.$$

The initial angular velocity and the initial magnetic field are both zero, so the final answer is

$$\boxed{\omega(t) = -\frac{qB(t)}{2m}.}$$

The minus sign implies that the angular velocity is directed opposite to the magnetic field.

Experimental Exam

Problem 1. Efficiency of a circuit.

Equipment:

DC voltage source (rectifier), 3 multimeters, resistor substitution box R_x, two resistors R_1 and R_2 in series, 8 wires, graph paper. See Figure 3.

Figure 3

Let us define the efficiency of the circuit on Figure 4 as the ratio of the power dissipated to the right of the voltmeter (i.e. at the resistor R_x and the ammeter I_x) to the total power dissipated in the circuit. The aim of this problem is to study the dependence of η on the load R_x (which will be calculated as $R_x = U_x/I_x$). This will be used to find the optimal value $R_{x,\mathrm{opt}}$ which maximises η (accurate to $5\,\Omega$), as well as the maximum η itself.

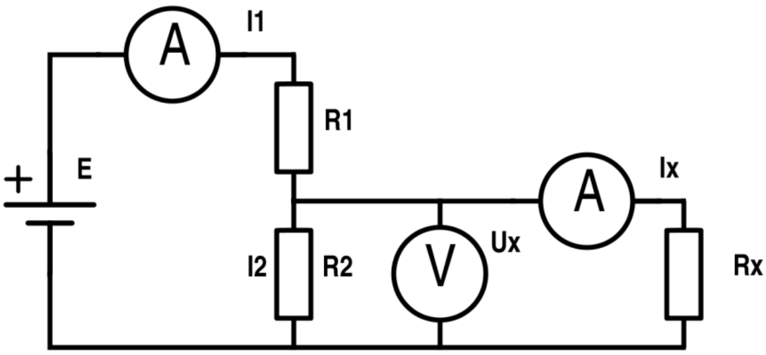

Figure 4

(a) Using a multimeter in ohmmeter mode, measure the resistances R_1 and R_2 and record them in a table. Turn on the voltage source in DC mode and set the voltage to $E = 2.00\,\mathrm{V}$. Leave the source on and do not change the voltage from now on. **(1.0 pt)**

(b) Assemble the circuit. Before connecting it to the voltage source, ask the examiner to confirm. **(3.0 pt)**

(c) Assume that the dependence $\eta(R_x)$ has a single maximum. Choose appropriate values for R_x in the range $(0.1R_x - 10R_x)$. Take the necessary measurements and tabulate the dependence for this range. In the table you should record both the nominal resistance of the substitution box and its measured value $R_x = U_x/I_x$. **(5.0 pt)**

(d) Plot a graph of $\eta(R_x)$. **(2.0 pt)**

(e) If your experimental data is insufficient, take additional measurements. If the scale of your graph was unsuitable, plot an additional graph which covers only the important features of the dependence. Write down your values for $R_{x,\text{opt}}$ and $\eta(R_{x,\text{opt}})$. **(4.0 pt)**

Call the examiner in case of any technical difficulties.