Theoretical Exam

Problem 4. A layer of tap water is poured into a wide and shallow dielectric vessel. The thickness of the layer is $h = 1.00\,\mathrm{cm}$. The probes of a multimeter are vertically submerged at a distance $l = 20.0\,\mathrm{cm}$ from each other until they reach the bottom of the vessel. The multimeter measures a resistance of $R = 500\,\mathrm{k\Omega}$. Assuming the probes are cylinders of radius $a = 1.00\,\mathrm{mm}$ and modelling the water as a weakly conducting medium, find a formula for the resistivity of water ρ and calculate its value.

Solution. This problem is about current flow through a continuous medium, in this case an infinite plane of thickness h. To find an expression for the resistance R, we need to see how the potential difference between the probes U depends on the current I that flows into the plane at one of the probes and gets drawn out at the other.

First, let us figure out what the current distribution actually looks like. If we only had I flowing in and nothing was going on at the other probe, the answer would be that current flows out into the plane radially, the current density being $\mathbf{j}_1 = \frac{I}{2\pi hr}\hat{\mathbf{r}}_1$. Likewise, if we were just drawing out I at the second probe, the answer is the same distribution, but this time directed inwards at the second probe: $\mathbf{j}_2 = -\frac{I}{2\pi hr}\hat{\mathbf{r}}_2$. Now, the key point is that the superposition $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$ will satisfy all the boundary conditions imposed in the actual problem – it corresponds to I flowing in at probe 1, and also to I being drawn out at probe 2, and there are no currents at infinity. This $\mathbf{j}$ adheres to the continuity equation (the counterpart of Kirchhoff's junction rule in the plane) because that equation is linear, and the two components $\mathbf{j}_1$ and $\mathbf{j}_2$ both comply with it. To sum up, $\mathbf{j}$ is a perfectly valid solution. We then exploit uniqueness to state that this is *the* solution.

When calculating the potential difference between the probes, we will work along the straight line that connects them. By Ohm's law, the electric field at a distance r from probe 1 is

$$\mathbf{E} = \frac{I\rho}{2\pi h}\left(\frac{1}{r} + \frac{1}{l-r}\right)\hat{\mathbf{r}}_1.$$

We integrate this expression between the surfaces of the two conducting cylinders, and this results in

$$U = \int_a^{l-a} \frac{I\rho}{2\pi h}\left(\frac{1}{r} + \frac{1}{l-r}\right)\mathrm{d}r \quad \Rightarrow \quad U = \frac{I\rho}{\pi h}\ln\left(\frac{l-a}{a}\right).$$

The multimeter displays $R = U/I$, so

$$\rho = \frac{R\pi h}{\ln\left(\frac{l-a}{a}\right)} \approx \boxed{\frac{R\pi h}{\ln\left(\frac{l}{a}\right)} = 3000\,\Omega\,\mathrm{m}.}$$

Problem 5. A square frame of side a and resistance R is placed next to an infinite conducting wire carrying a current I. The wire lies in the plane of the frame and is parallel to two of its sides. The distance from the wire to the near end of the frame is d ($d \gg a$). The frame starts to move away from the wire with constant velocity v (in the plane of the frame and perpendicular to the wire). Find a formula for the total heat Q dissipated at the frame until it is infinitely far from the wire. Calculate Q for $a = 2\,\mathrm{cm}$, $R = 0.01\,\Omega$, $d = 20\,\mathrm{cm}$, $I = 10\,\mathrm{A}$, $v = 5\,\mathrm{cm/s}$. The inductance of the frame is negligible.

Solution. The magnetic field at a distance r from the wire is $B = \frac{\mu_0 I}{2\pi r}$, and it crosses the frame perpendicularly. When the near end of the frame is at r, the flux through the frame would be

$$\Phi = \int_r^{r+a} \frac{\mu_0 I a}{2\pi r}\mathrm{d}r = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{r+a}{r}\right).$$

As the frame recedes with velocity v, the flux changes, and there's an induced EMF

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mu_0 I a}{2\pi}\left(\frac{r}{r+a}\right)\left(\frac{a}{r^2}\right)v.$$

This means there's a current $I = \mathcal{E}/R$, and so the instantaneous power dissipation is $P = \mathcal{E}I = \mathcal{E}^2/R$. Now it's time to use $r \gg a$. We reach

$$P = \left(\frac{\mu_0 I v a^2}{2\pi}\right)^2\left(\frac{1}{R}\right)r^{-4}.$$

We know that $r = d + vt$, and all that's left is to integrate, $Q = \int_0^\infty P\,\mathrm{d}t$. The answer is

$$\boxed{Q = \frac{\mu_0^2 I^2 a^4 v}{12\pi^2 R d^3} = 1.3 \times 10^{-16}\,\mathrm{J}.}$$

Problem 6. A monochromator consists of:

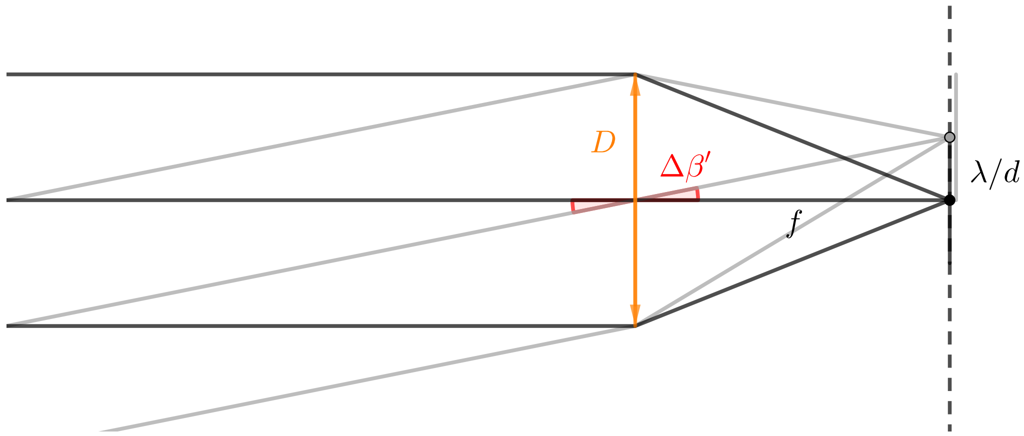
- An isosceles glass prism of base $b = 10.0\,\mathrm{cm}$ and an angle $\theta = 50^\circ$ opposite the base.
- A thin entrance slit.
- An identical exit slit.
- A lens at the entrance which gives rise to a parallel beam incident on the prism.
- An identical lens which focuses the dispersed light onto the exit slit. The lenses are of diameter $D = 10.0\,\mathrm{cm}$ and focal length $f = 50.0\,\mathrm{cm}$.

The glass of the prism has a refractive index $n = 1.73$. Its dispersion at wavelength 550 nm is $\frac{\mathrm{d}n}{\mathrm{d}\lambda} = 1.35 \times 10^5\,\mathrm{m}^{-1}$. Inside the prism light travels parallel to the base. Using the Rayleigh criterion, find the theoretical upper limit for the spectral resolution of the monochromator $R = \lambda/\Delta\lambda$ (due to diffraction). Calculate R.

Solution. This is quite tough, and we should try to get our head around it using a diagram. First, the entrance slit has to be positioned at the focus of the first lens. This ensures that all the light that emerges from the slit will come out as a parallel beam of diameter D. The beam is then incident on the prism, with the angle of incidence α set up so that the rays will be parallel to the base of the prism following their refraction. The angle of refraction should be equal to $\theta/2$ for this to happen, so the angle of incidence can be found from $\sin\alpha = n\sin(\theta/2)$. Upon leaving the prism, the rays undergo refraction a second time, and come out as a parallel beam by symmetry. This beam is incident on the second lens, which focuses it onto a single point in its focal plane. This is where the exit slit is.

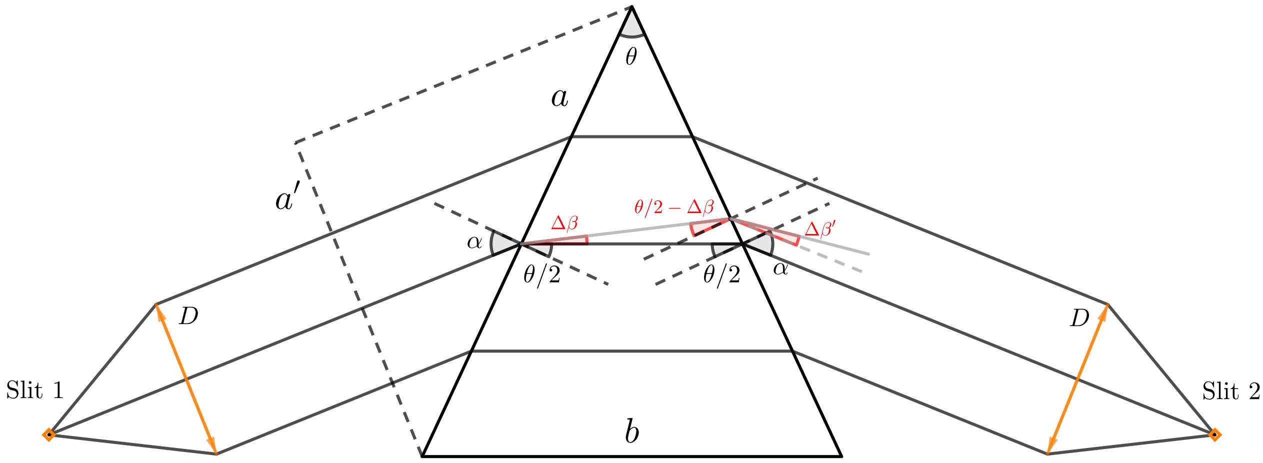
So far we have outlined how the prism works at a fixed wavelength, say $\lambda_0 = 550\,\mathrm{nm}$. However, the prism's refractive index varies with λ, and light whose wavelength is shifted from λ_0 by some $\Delta\lambda$ will encounter a refractive index shifted by $\Delta n = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\Delta\lambda$. Such rays will take on a different trajectory, but they will again come out as a parallel beam. Each λ will then have its own parallel beam which is focused at a specific point in the focal plane of the second lens. This would imply that all the different wavelengths can be distinguished by their position in the focal plane at the exit slit. Unfortunately, we also have diffraction at the second lens, which means that the focusing is imperfect. Instead of a single point, we get a smeared-out image with a characteristic angular size $\chi \approx 2\lambda/d$, where d is the relevant aperture at the second lens. This can either be the diameter of the lens D or the width of the incident beam l, if that's less than D – we'll need to check this later. Now, according to the Rayleigh criterion, two wavelengths will be indistinguishable if their spots overlap too much, the limit being when the centre of one spot lies on the rim of the other. Let the parallel beams for two distinct wavelengths make an angle $\Delta\beta'$ when they come out of the prism. As we can see from the diagram, the distance between the centres of their spots at the exit slit is $\Delta\beta' f$, while the 'radius' of each spot is

$\frac{\lambda f}{d}$. Hence, the limiting $\Delta\beta'$ is equal to λ/d. We will need to express this $\Delta\beta'$ in terms of the wavelength difference $\Delta\lambda$, after which we can write $R = \lambda/\Delta\lambda$, and we're done.



Consider two rays that diverge due to a difference Δn in the refraction index. Since both are incident at the same α, we can differentiate Snell's law to find the difference $\Delta\beta$ in their refraction angles, like so:

$$n \sin\beta = \sin\alpha \quad \Rightarrow \quad \Delta n \sin\beta + n\cos\beta\,\Delta\beta = 0 \quad \Rightarrow \quad \Delta\beta = -\frac{\Delta n}{n}\tan\beta.$$



When the rays get to the other side of the prism, the difference in their incidence angles is $\Delta\alpha' = -\Delta\beta$. After they exit the prism, their refraction angles will differ by some $\Delta\beta'$ which can be obtained from Snell's law. This time around we need to be careful, since both the incidence angle and the refractive index will vary:

$$n\sin\alpha' = \sin\beta' \quad \Rightarrow \quad \Delta n \sin\alpha' + n\cos\alpha'\,\Delta\alpha' = \cos\beta'\Delta\beta'$$

Now we can substitute $\alpha' = \beta = \frac{\theta}{2}$. After plugging in our expression for $\Delta\beta$, the left hand side adds up to $2\Delta n \sin\frac{\theta}{2}$. As for the right hand side, we can use $\beta' = \alpha$ and $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-n^2\sin^2\left(\frac{\theta}{2}\right)}$. Then we have

$$\Delta\beta' = \frac{2\sin\left(\frac{\theta}{2}\right)\left(\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right)\Delta\lambda}{\sqrt{1-n^2\sin^2\left(\frac{\theta}{2}\right)}}.$$

Next, let's figure out the aperture d. The legs of the prism have length $a = \frac{b/2}{\sin(\theta/2)}$, and their projection perpendicular to the incident beam is

$$a' = a\cos\alpha = \frac{b}{2\sin\left(\frac{\theta}{2}\right)}\sqrt{1 - n^2\sin^2\left(\frac{\theta}{2}\right)} = 8.1\,\mathrm{cm}.$$

This is less than $D = 10.0\,\mathrm{cm}$, so the transmitted beam's cross-section will be limited to a' along some direction. The effective aperture is then $d = a'$. Therefore,

$$R = \frac{\lambda}{\Delta\lambda} = \frac{\Delta\beta' d}{\Delta\lambda} = \frac{\Delta\beta' a'}{\Delta\lambda} = \boxed{b\frac{\mathrm{d}n}{\mathrm{d}\lambda} = 13500.}$$

Note that we have diffraction at the first lens and at the prism as well, but this doesn't matter much by the time we get to the second lens. For a similar problem about dispersion, see RMPh 2017-1A.

Experimental Exam

Problem 1. Ruler vibrations.

Equipment:
Ruler (50 cm), another ruler (shorter), clamp, wooden block, 3 old coins (Bulgarian, issue 1992, one is 5 leva, the other two are 1 lev), stopwatch, tape measure, Scotch tape, scissors, graph paper.

Data:

Mass of the ruler	m_L	51.0 g
Thickness of the ruler	h	3.00 mm
Mass of the 5 leva coin	m_5	6.00 g
Mass of a 1 lev coin	m_1	4.00 g

You will need to complete five tasks.

(a) Find the density of the plastic ρ that the ruler is made from. The ruler's cross section can be approximated as a parallelogram. **(1.5 pt)**

Clamp one end of the ruler so that the clamped part is horizontal and the free part is $L_0 = 47.0\,\text{cm}$ long. Load the ruler at its free end using different sets of coins, taking care that the centre of mass of the coins is exactly at the end of the ruler. Measure the additional deflection of the ruler s due to the coins. This deflection is given by

$$s = \frac{4FL_0^3}{Ebh},$$

where F is the force at the end of the ruler which causes the deflection, L_0 is the length of the ruler, b is its width, h is its length, and E is the Young modulus of the plastic. This formula applies for a rectangular cross section, but it is accurate enough in the case of our ruler as well.

(b) Plot a graph of the deflection s against the mass of the coins m. Using the graph, find the Young modulus of the plastic E. **(3.0 pt)**

Study the dependence of the oscillation period T of the free end of the ruler (without loading it with coins) on the free length L. This dependence is described by the formula

$$T = \frac{CL^2}{h}\sqrt{\frac{\rho}{E}},$$

where L is the free length, h is the thickness of the ruler, and C is some unknown constant.

(c) Plot a linearised graph in the appropriate variables. Using the graph, find C. Use the values of ρ and E from the previous parts of the problem. **(4.5 pt)**

Clamp the ruler at one end so that it is vertical, with the part that sticks out being $L_0 = 47.0\,\text{cm}$ long. Attach different sets of coins at its free end, taking care that the centre of mass of the coins is exactly at the end of the ruler.

(d) Study the dependence of the oscillation period T of the free end of the ruler on the total mass of the attached coins m. Plot your data in a way that will allow you to predict the period T_{20} for $m = 20\,\text{g}$ by extrapolating. **(4.0 pt)**

(e) Find the critical mass M at which the equilibrium of the vertical ruler becomes unstable and the ruler stops oscillating. **(2.0 pt)**

Call the examiner in case of any technical difficulties.

Problem 2. Induction brake.

Introduction:

A permanent magnet moving along a non-magnetic metal surface will induce eddy currents in the metal. These give rise to a magnetic field that results in a drag force $\mathbf{F}$ on the magnet:

$$\mathbf{F} = -k\mathbf{v},$$

where $\mathbf{v}$ is the velocity of the magnet and k is the magnetic drag coefficient. This coefficient depends on the shape and size of the magnet, the magnetic field of the magnet, and the conductivity of the metal surface. The aim of this problem is to find k for a magnet moving along an inclined aluminium surface.

Equipment:

1. Aluminium rail with U-shaped cross section (Figure 1).
2. Neodymium magnet of mass 5.6 g in the shape of a rectangular cuboid. It is magnetised parallel to its shortest edge, i.e. its poles are on its largest surfaces.
3. Stopwatch.
4. Plastic tape measure, accurate to 1 mm.
5. Pencil.
6. Ball of plasticine.
7. Ruler.
8. Blank paper and two sheets of graph paper.

Note: The magnet is made from a brittle alloy and will easily break if struck. Take care not to break it.

Tasks:

(a) Write down the equation of motion for a magnet on a metal surface at an angle α to the horizon. The magnetic drag coefficient is k and the coefficient of dynamic friction is μ. Find an expression for the terminal velocity of the magnet v_{t} assuming the rail is long enough.

(b) Place the rail an an angle of 60° to the horizontal table. Explain how you have found the angle. You can affix the lower end of the rail to the plasticine and lean the upper end on the wall. Place the magnet with one of its poles lying on the aluminium surface. Let the magnet slide with zero initial velocity from different initial heights. Study the dependence of distance moved s against time t. Present your data in tabular and graphical form.

(c) Analyse the graph from (b). State the type of motion within the time intervals you worked with. Find the terminal velocity of this motion v_{t}.

(d) Study the descent of the magnet at different slopes $\alpha < 60°$ (though large enough for the magnet to slide). Find the terminal velocity v_{t} for each of the angles α. Present your results in a table.

(e) Find auxiliary variables x and y (expressed in terms of the slope α and the terminal velocity v_{t}) for which a linear dependence is expected. Present your data in terms of these variables in tabular and graphical form. Using your graph, find the coefficient of dynamic fricion μ and the magnetic drag coefficient k.

Figure 1

(f) The time dependence for the velocity of the magnet is given by

$$v(t) = v_{\mathrm{t}}(1 - e^{-t/\tau}), \tag{1}$$

where τ is a time constant which is independent of the slope (that is, the time taken to reach 63.2 % of the terminal velocity). Using your data, find the time constant τ. Is it possible to verify Equation (1) experimentally using only the equipment given?