Short Exam 1

Problem. A thin rod of mass m and length L is placed vertically on a horizontal surface. The acceleration due to gravity is g. The rod is given a tiny lateral push and it starts falling. There is no friction between the rod and the surface. Right before the rod strikes the surface horizontally, find:

(a) The velocity of the centre of mass V_C.

(b) The angular acceleration of the rod ε.

(c) The normal force R.

Solution. (a) The motion of an object can always be decomposed into translation of the centre of mass (CM) and rotation about the CM. In this problem, the CM doesn't move horizontally because the only external forces are the normal force R and the gravity mg, both of which are vertical.

Denote the CM velocity by V and the angular velocity by ω. We will work at an instant when the rod makes an angle θ with the horizon, and we'll take the special case $\theta = 0$ when needed. Consider the contact point between the rod and the floor. Its vertical velocity is always zero, which can only happen when the rotation cancels the CM velocity, such that $V = \frac{\omega L}{2}\cos\theta$. Also, there are no frictional losses, so the total energy is conserved. We will compare the initial state when the CM is at $y = \frac{L}{2}$ and the current state, for which $y = \frac{L}{2}\sin\theta$:

$$mg\frac{L}{2} = \frac{I\omega^2}{2} + \frac{mV^2}{2} + mg\frac{L}{2}\sin\theta.$$

Here the moment of inertia of the rod is $\frac{1}{12}mL^2$. Substituting our expression for ω, we find

$$\frac{1}{2}mV^2\left(1 + \frac{1}{3\cos^2\theta}\right) = mg\frac{L}{2}(1-\sin\theta) \quad \Rightarrow \quad V = \sqrt{gL\left(\frac{1-\sin\theta}{1+\frac{1}{3\cos^2\theta}}\right)}.$$

When the rod is horizontal, we answer

$$V_C = V(0) = \boxed{\sqrt{\frac{3}{4}gL}.}$$

(b)-(c) Consider the contact point at the instant when the rod is horizontal. In its motion *about the centre of mass* it has some normal acceleration, which is horizontal, and some tangential acceleration, which is vertical. Because the vertical velocity of this point is always zero, the same can be said for its vertical acceleration. But that is precisely equal to the tangential acceleration minus the CM acceleration a_{CM}. We conclude that

$$a_{\text{CM}} = \frac{\varepsilon L}{2}.$$

Apart from that, we have the translational equation of mtion

$$ma_{\text{CM}} = mg - R,$$

Finally, only the normal force will generate a torque about the centre of mass, so we have

$$R\frac{L}{2} = \frac{mL^2}{12}\varepsilon.$$

Solving this set of equations,

$$\boxed{R = \frac{mg}{4}}, \qquad \boxed{\varepsilon = \frac{3g}{2L}.}$$

If you don't have the relation $a_{\text{CM}} = \frac{\varepsilon L}{2}$, the problem is still doable, at the cost of more algebra. Working in the general case, the angular velocity is

$$\omega = \frac{2}{L\cos\theta}V = 2\sqrt{\frac{3g}{L}}\sqrt{\frac{1-\sin\theta}{1+3\cos^2\theta}}.$$

The acceleration of the CM will be

$$a_{\text{CM}} = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{2}\sqrt{gL}\sqrt{\frac{1+\frac{1}{3\cos^2\theta}}{1-\sin\theta}} \cdot \frac{(-\cos\theta)\left(1+\frac{1}{3\cos^2\theta}\right) - (1-\sin\theta)\left(-\frac{2}{3}\frac{-\sin\theta}{\cos^3\theta}\right)}{\left(1+\frac{1}{3\cos^2\theta}\right)^2}\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

By using that $\omega = -\frac{\mathrm{d}\theta}{\mathrm{d}t}$, this simplifies to

$$a_{\text{CM}} = \frac{(\cos\theta)\left(1+\frac{1}{3\cos^2\theta}\right) + \frac{2}{3}\frac{(1-\sin\theta)\sin\theta}{\cos^3\theta}}{\left(1+\frac{1}{3\cos^2\theta}\right)^2}g.$$

When $\theta = 0$, this corresponds to $a_{\text{CM}} = \frac{3}{4}g$. From that point on, use

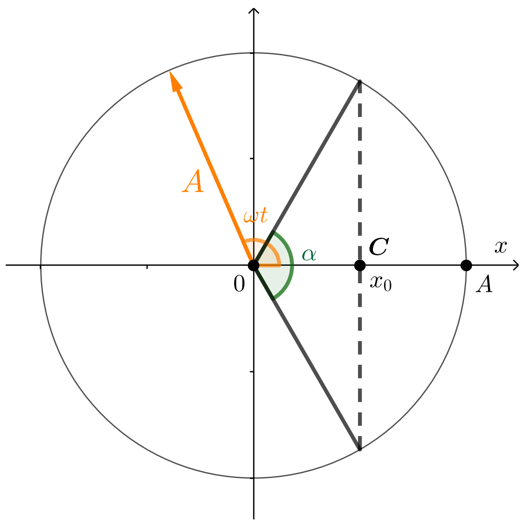
$$ma_{\text{CM}} = mg - R \quad \text{and} \quad R\frac{L}{2} = \frac{mL^2}{12}\varepsilon,$$

and you get the same final answers.

Theoretical Exam

Problem 1. A point mass oscillates harmonically along a line. The mass passes through a point C of its trajectory in alternating time intervals of 1 s, 2 s, 1 s, 2 s, ... What is the ratio of the distances between C and the endpoints of the trajectory?

Solution. Let the motion have an amplitude A and angular frequency ω. Without loss of generality, we can describe it with $x = A\cos(\omega t)$. Let the position of point C be x_0. We want to find x_0/A. A clean way to do this is to note that x can be interpreted as the projection along the x-axis of a vector with magnitude A which rotates counterclockwise with an angular frequency ω:



Between two consecutive crossings of C the vector rotates either by α or $2\pi - \alpha$. Assuming $\alpha < \pi$, we know that

$$\frac{\alpha}{2\pi - \alpha} = \frac{1\,\mathrm{s}}{2\,\mathrm{s}}.$$

Then $\alpha = \frac{2\pi}{3}$. On the other hand, $\cos\frac{\alpha}{2} = \frac{x_0}{A}$, so $\frac{x_0}{A} = \frac{1}{2}$. The distances between C and the endpoints are $A - x_0$ and $A + x_0$, so the ratio is

$$q = \frac{A - x_0}{A + x_0} = \frac{1 - (x_0/A)}{1 + (x_0/A)} = \boxed{\frac{1}{3}}.$$

Problem 2.

(a) Find the moment of inertia of a rectangle of mass m and sides a and b with respect to an axis passing through its centre of mass perpendicularly to its plane. **(1.0 pt)**

(b) A sheet of size AN is defined as a rectangle of surface area $2^{-N}\,\mathrm{m}^2$ and a ratio of $\sqrt{2}$ between its sides. Calculate the oscillation period of a vertical A4 sheet about a horizontal axis passing perpendicularly to the sheet through the middle of its longer side. Your answer should be accurate to 3 significant figures. **(2.0 pt)**

Solution. (a) The moment of inertia about the centre of mass is

$$\sum \Delta m(x^2 + y^2),$$

where $x \in \left[-\frac{a}{2}; +\frac{a}{2}\right]$ and $y \in \left[-\frac{b}{2}; +\frac{b}{2}\right]$. Along each axis, the mass distribution is the same as that of a uniform rod. So, from the x^2 term we will get a contribution of $\frac{1}{12}ma^2$, and likewise from the y^2 term we have $\frac{1}{12}mb^2$. The answer is $\boxed{I = \frac{1}{12}m(a^2 + b^2)}$.

(b) If the shorter side of sheet is b, the longer one is $\sqrt{2}b$. Using the parallel axis theorem, the moment of inertia with respect to the pivot is

$$I' = I + m\left(\frac{b}{2}\right)^2 = \frac{1}{12}m\left(a^2 + 4b^2\right) = \frac{1}{2}mb^2.$$

The sheet is a physical pendulum with a distance of $b/2$ between the pivot and the centre of mass. Its torque equation is

$$I'\ddot{\theta} + mg\left(\frac{b}{2}\right)\theta = 0,$$

which describes oscillations with a period of $T = 2\pi\sqrt{\frac{2I}{mgb}} = 2\pi\sqrt{\frac{b}{g}}$. Now, the length of b for a sheet of size AN can be found from $(2^{1/2}b)b = 2^{-N}\mathrm{m}^2$, which comes out as $b = 2^{-\frac{1}{2}\left(N+\frac{1}{2}\right)}\,\mathrm{m}$. For $N = 4$ the period is then $\boxed{T = 0.920\,\mathrm{s}}$.

Problem 3. A particle of mass m and charge q moves under a constant magnetic field B and an alternating electric field $E(t) = E_m\cos(\omega t)$, where $\omega = \frac{qB}{m}$. At time $t = 0$ the particle is at rest at the centre of the coordinate system (Figure 1). Find the classical equations of motion of the particle $x(t)$, $y(t)$, and $z(t)$. Describe the shape of the trajectory.

Hint: The motion along Oy is described by a function of the form $y(t) = At\sin(\omega t)$.

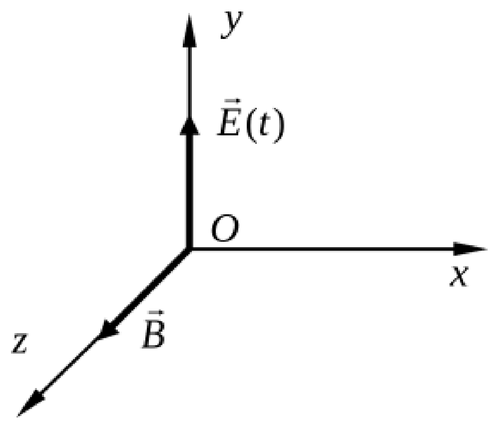


Figure 1

Solution. This problem is nearly identical to Problem 5 from 2011, though a bit heavier on the calculations. To start with, we will note that the total force $\mathbf{F} = q\mathbf{v} \times \mathbf{B} + q\mathbf{E}$ is constrained to the xy-plane, and because the particle is initially at rest, it will always remain in that plane. Thus, $\boxed{z(t) = 0.}$

The components of the force in the xy-plane are

$$F_x = m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qv_yB,$$
$$F_y = m\frac{\mathrm{d}v_y}{\mathrm{d}t} = qE_m\cos(\omega t) - qv_xB.$$

The hint tells us something about $y(t)$, so we would like to obtain an equation in v_y only. We differentiate the second equation and then refer to the first one to reach

$$\frac{\mathrm{d}^2v_y}{\mathrm{d}t^2} + \omega^2 v_y = -\omega^2\left(\frac{E_m}{B}\right)\sin(\omega t).$$

Let us work with $y(t) = A_1(\omega t)\sin(\omega t)$, where $A_1\omega = A$. It's a bit easier to take derivatives with respect to (ωt) rather than t. Using $\frac{\mathrm{d}}{\mathrm{d}t} = \omega\frac{\mathrm{d}}{\mathrm{d}(\omega t)}$, we get

$$v_y = A_1\omega(\sin(\omega t) + (\omega t)\cos(\omega t)),$$
$$\dot{v}_y = A_1\omega^2(2\cos(\omega t) - (\omega t)\sin(\omega t)),$$
$$\ddot{v}_y = A_1\omega^3(-3\sin(\omega t) - (\omega t)\cos(\omega t)).$$

Matching both sides of the differential equation, we see that

$$-2A_1\omega^3\sin(\omega t) = -\omega^2\left(\frac{E_m}{B}\right)\sin(\omega t),$$

or $A = A_1\omega = \frac{E_m}{2B}$. As a result, $\boxed{y(t) = \frac{E_m}{2\omega B}(\omega t)\sin(\omega t).}$

Now we turn to $x(t)$. Having found A, we can get v_x from the F_y equation:

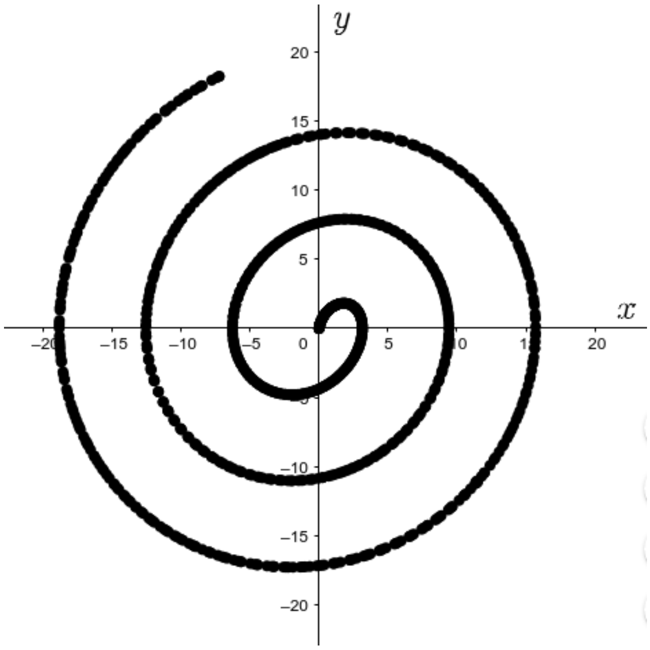
$$v_x = \frac{E_m}{B}\cos(\omega t) - \frac{1}{\omega}\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{E_m}{2B}(\omega t)\sin(\omega t).$$

Finally, integrating this with $x(0) = 0$ gives us

$$\boxed{x(t) = \frac{E_m}{2\omega B}\left(-(\omega t)\cos(\omega t) + \sin(\omega t)\right).}$$

You should double-check that the initial conditions $x(0) = 0$, $y(0) = 0$, $v_x(0) = 0$, and $v_y(0) = 0$ are all satisfied. You could also replace ω with $\frac{qB}{m}$ in the final answers if you feel like it.

The trajectory is a $\boxed{\text{spiral}}$, as shown. In the equation for $x(t)$ there is a harmonic term $\sim \sin(\omega t)$ which becomes negligible as $\omega t \to \infty$, when the other terms will blow up. Those terms correspond to an Archimedean spiral $r(\varphi) = A\varphi$. It is fair to say that our trajectory asymptotes to an Archimedean spiral.



Experimental Exam

Problem 1. Measuring the refractive index of a liquid.

Equipment:
Plastic petri dish (shallow cylindrical dish for cell culture use), unknown liquid, pins, piece of corrugated cardboard, ruler, graph paper, pens (3 colours). See Figure 2.

Figure 2

The aim of this problem is to find the refractive index of an unknown liquid through the appropriate measurements. Do not taste the liquid! Though non-toxic, it has strong laxative properties[1]!

Fill the the petri dish with the liquid and place it on top of the graph paper and the corrugated cardboard. Use the pins to mark the incident and the outgoing rays. Observe the images of the pins through the side of the petri dish. Study the dependence of the angle of deviation φ on the angle of incidence α (Figure 2). These angles are related by

$$\sin\alpha = n\sin\left(\alpha - \frac{\varphi}{2}\right).$$

Assume that the plastic has the same refractive index as the liquid.

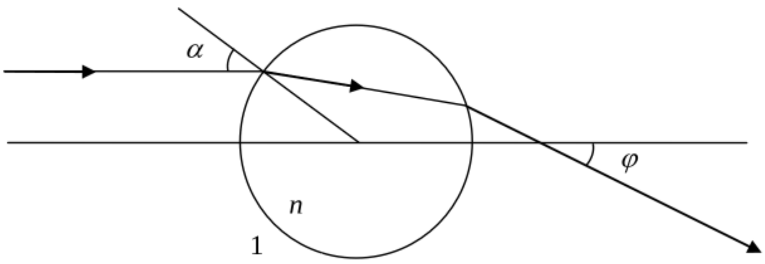


Figure 3

(a) State the parameters that you will be measuring. Take enough useful measurements. Present them in a table and explain how they were obtained. **(6.0 pt)**

(b) State the variables which, when plotted, can easily give you n. **(0.5 pt)**

(c) Plot the relevant graph. **(4.5 pt)**

[1] This turned out to be a lie, as the liquid was just a sugar solution.

(d) Using the graph, determine the refractive index n. **(3.0 pt)**

(e) Estimate your error in finding n. **(1.0 pt)**

Call the examiner in case of any technical difficulties.