Short Exam 2

Problem. A circuit consists of a battery of EMF E, a capacitor C, a resistor R, and an open switch, all in series. At time $t = 0$ the switch is closed.

(a) Find the magnitude $I_1(0)$ of the initial current, the time dependence of the current $I_1(t)$, and the total heat Q_1 dissipated at the resistor until the capacitor is fully charged. **(2.0 pt)**

The capacitor is discharged and the resistor is replaced by a nonlinear element with an I-V curve $I = \beta U^{3/2}$, where β is a constant. At time $t = 0$ the switch is closed.

(b) Find the magnitude $I_2(0)$ of the initial current, the time dependence of the current $I_2(t)$, and the total heat Q_2 dissipated at the nonlinear element until the capacitor is fully charged. **(3.0 pt)**

Solution. (a) Initially the charge q on the capacitor is zero, likewise for its voltage. The voltage loop rule for the circuit is

$$E - \frac{q}{C} - IR = 0.$$

Initially we just have $E - I_1(0)R = 0$, or $\boxed{I_1(0) = \frac{E}{R}.}$ In general, $I = \frac{\mathrm{d}q}{\mathrm{d}t}$, so

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \left(\frac{1}{RC}\right) q = \frac{E}{R}.$$

The general solution is $q(t) = Ae^{-\frac{t}{RC}} + CE$, where A is an arbitrary constant. Using the initial condition $q(0) = 0$, we obtain

$$q(t) = CE\left(1 - e^{-\frac{t}{RC}}\right) \quad \Rightarrow \quad \boxed{I(t) = \left(\frac{E}{R}\right) e^{-\frac{t}{RC}}.}$$

We can now find the dissipated heat Q_1 by integrating $\int_0^\infty I^2 R \, \mathrm{d}t$, but the algebra can be avoided. The point is that Q_1 is equal to the total energy input from the battery, minus the energy stored in the capacitor in the final state (when its charge is $q_0 = CE$). Then,

$$Q_1 = q_0 E - \frac{q_0^2}{2C} = \boxed{\frac{CE^2}{2}.}$$

(b) Let the voltage on the nonlinear element be U_N, so that the current in the circuit is $I = \beta U_N^{3/2}$. As in the previous part, initially $q/C = 0$, so $U_N = E$, and therefore $\boxed{I_2(0) = \beta E^{3/2}.}$ The general voltage loop rule is

$$E - \frac{q}{C} - U_N = 0.$$

We want to involve the current I, so we differentiate this equation to get

$$-\frac{I}{C} - \frac{\mathrm{d}U_N}{\mathrm{d}t} = 0.$$

Now $U_N = (I/\beta)^{2/3}$, hence

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\left(\frac{3\beta^{2/3}}{2C}\right) I^{4/3}.$$

After integrating, we are left with

$$I(t) = \left(B + \frac{\beta^{2/3}}{2C}t\right)^{-3},$$

where B is a constant. After imposing the initial condition for the current,

$$\boxed{I(t) = \beta\left(\frac{1}{\sqrt{E}} + \frac{\beta t}{2C}\right)^{-3}.}$$

The dissipated heat can now be found by integrating $\int_0^\infty U_N I \, \mathrm{d}t$, but we can save ourselves the work by using the same trick as before. The answer is, again, $\boxed{Q_2 = \frac{CE^2}{2}.}$

Theoretical Exam

Problem 4. We wish to make a solenoid of inductance $L = 1\,\mathrm{mH}$ and length $l = 1\,\mathrm{m}$. The diameter of the solenoid is $d \ll l$.

(a) Find the length of the wire x that we will need. **(1.5 pt)**

(b) If the wire is made of copper and has a resistance of $R = 1.7\,\Omega$, find the mass of the solenoid m. **(1.5 pt)**

The density of copper is $\rho_m = 8.9\,\mathrm{g/cm^3}$ and its resistivity is $\rho_R = 17\,\mathrm{n\Omega\,m}$.

Solution. (a) The inductance of the solenoid is $L = \Phi/I$, where Φ is the magnetic flux through its interior when it carries current I. Let the solenoid have N turns. From Ampère's circuital law, the magnetic field inside the solenoid is approximately uniform and equal to $\frac{\mu_0 NI}{l}$. The flux through a single turn is $\frac{\mu_0 NI}{l}\frac{\pi d^2}{4}$, but there are N of those, so the total flux is $\Phi = \frac{\mu_0 N^2 I}{l}\frac{\pi d^2}{4}$. The inductance is then $L = \frac{\mu_0 N^2}{l}\frac{\pi d^2}{4}$. At the same time, the total length of the wire is $x = N\pi d$. We can express this length as

$$\boxed{x = \sqrt{\frac{4\pi l L}{\mu_0}} = 100\,\mathrm{m}.}$$

(b) Let the cross-section of the wire be S_w. The resistance and mass of the wound wire are respectively given by

$$R = \frac{\rho_R x}{S_w} \quad \text{and} \quad m = \rho_m S_w x.$$

We multiply these to cancel S_w, and we get

$$\boxed{m = \frac{4\pi \rho_m \rho_R l L}{\mu_0 R} = 0.89\,\mathrm{kg}.}$$

Problem 5. A particle of mass m and charge q is located at the origin of a Cartesian coordinate system at time $t = 0$. Around the particle there is a constant homogeneous magnetic field B along Oz and an oscillating homogeneous electric field $E(t) = E_0 \sin(\omega t)$ along Ox, with $\omega = \frac{qB}{m}$. Assume that the particle moves along an Archimedean spiral given by $r(\varphi) = A\varphi$ in polar coordinates. Here A is a constant and φ is measured starting from Ox. If the particle moves with a constant angular velocity ω, find the increase in the radius vector per turn.

Solution. Let $\mathbf{E}(t) = E_0 \sin(\omega t)\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$. The total force on the particle is $\mathbf{F} = q\mathbf{v} \times \mathbf{B} + q\mathbf{E}$, which has components

$$F_x = m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qE_0 \sin(\omega t) + qv_y B$$
$$F_y = m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -qv_x B.$$

We differentiate the equation for F_x and then substitute the result for $\frac{\mathrm{d}v_y}{\mathrm{d}t}$ from the F_y equation. This way we get an equation in v_x only:

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + \omega^2 v_x = \left(\frac{E_0}{B}\right)\omega^2 \cos(\omega t).$$

Now the idea is to find v_x, then get v_y, then integrate the velocities to obtain $x(t)$ and $y(t)$, and thus derive $r(t)$. However, the differential equation at hand is rather difficult. It represents driven harmonic oscillations at resonance, meaning that the driving term's frequency is the same as the natural frequency of the system ω. In this case, a trial solution of the sort $v_x = C\cos(\omega t + \theta)$ will not work. Instead, the problem statement wants us to try something else, namely $r(t) = A(\omega t)$, which corresponds to $x(t) = A(\omega t)\cos(\omega t)$ and[1]

$$v_x(t) = A\omega(\cos(\omega t) - (\omega t)\sin(\omega t)).$$

Our goal is to determine a constant A which satisfies the differential equation for v_x. To check this, we will also need the second derivative of v_x. Using the shortcut $\frac{\mathrm{d}v_x}{\mathrm{d}t} = \omega\frac{\mathrm{d}v_x}{\mathrm{d}(\omega t)}$, we can quickly find

$$\dot{v}_x = A\omega^2\left(-2\sin(\omega t) - (\omega t)\cos(\omega t)\right),$$
$$\ddot{v}_x = A\omega^3\left(-3\cos(\omega t) + (\omega t)\sin(\omega t)\right).$$

After we plug this into the differential equation, the nasty terms cancel, and we have

$$-2A\omega\cos(\omega t) = \left(\frac{E_0}{B}\right)\cos(\omega t),$$

from which we extract $A = -\frac{mE_0}{2qB^2}$. The trajectory of the particle is then

$$r(t) = -\frac{E_0 t}{2B}.$$

It is completely fine to have a negative $r(t)$. For example, if $r = -2\,\mathrm{m}$ at $\varphi = 30°$, this just means that the particle is at a distance of $2\,\mathrm{m}$ from the origin in a direction $\varphi' = 30° + 180° = 210°$ from the x-axis. Back to the problem at hand. In a single turn φ increases by 2π. Since the distance to the origin is $r = A\varphi$, the increase in the radius vector's magnitude per turn will be

$$\Delta r = 2\pi \cdot |A| = \boxed{\frac{\pi m E_0}{qB^2}}.$$

Problem 6. A parallel beam of monochromatic light (wavelength $\lambda = 500\,\mathrm{nm}$) is normally incident on a diffraction grating. The period of the grating is $d \gg \lambda$. A convex lens of focal length $f = 1\,\mathrm{m}$ is placed parallel to the grating. A diffraction pattern is then observed on a screen in the focal plane of the lens. We now rotate the diffraction grating at an angle α with respect to an axis parallel to the slits. The new fourth order maximum is now observed where the fifth order maximum originally was.

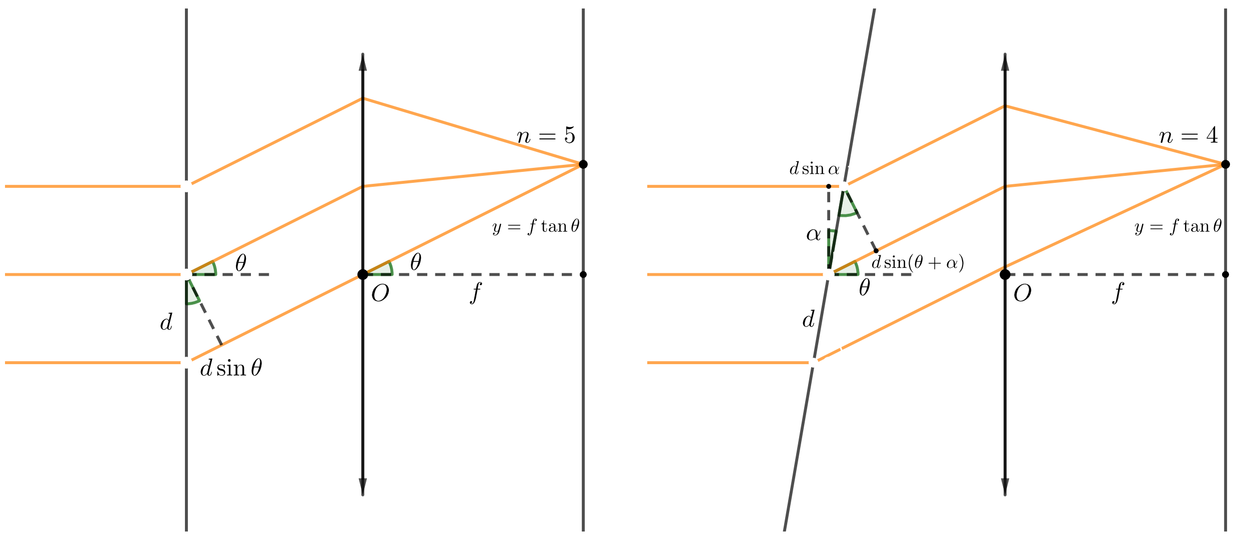
[1] Note that $v_x(0) \neq 0$. In the original problem statement the particle is initially at rest, which makes the problem overdetermined.

(a) Find the angle α. **(1.0 pt)**

(b) Given that the fifth order maximum has shifted by $\Delta l = 12.5\,\mathrm{mm}$ along the screen due to the rotation, find the period of the grating d in micrometres. **(1.0 pt)**

(c) The screen is $L = 20\,\mathrm{cm}$ wide and positioned symmetrically with respect to the optical axis. Find the number of maxima N that will be observed on the screen after the rotation. **(1.0 pt)**

Solution. (a) In the initial setup, all rays that diffract at an angle θ form a parallel beam which is focused into a single point on the screen (because the screen is in the focal plane). Any two rays from adjacent slits acquire an optical path difference of $d\sin\theta$ at the grating. Since there are many such pairs, the only way to get a maximum on the screen is for all rays to stay in phase, i.e. $d\sin\theta = n\lambda$ for integer n. For a given n, we call this the n-th order maximum.



After we rotate the grating by an angle α, the optical path difference for the beam at an angle θ will change to $d(\sin(\theta+\alpha) - \sin\alpha)$. As per the problem statement, for some angle θ_5 we have

$$d\sin\theta_5 = 5\lambda,$$
$$d(\sin(\theta_5+\alpha) - \sin\alpha) = 4\lambda.$$

Since $d \gg \lambda$, the angle θ_5 is expected to be very small. The same cannot be said for α, but we are still allowed to approximate $\sin\theta_5 + \alpha \approx \sin\alpha + \theta_5\cos\alpha$. Thus

$$d\theta_5 = 5\lambda,$$
$$d\theta_5\cos\alpha = 4\lambda.$$

We divide these to find $\cos\alpha = 4/5$, or $\boxed{\alpha = 36.9^\circ}$. For an exact value of α, we'd also need the data from (b). Working out a system of three equations numerically, we could obtain $\alpha = 35.5^\circ$. Evidently our approximate value for α isn't far from the exact answer.

(b) Let the new direction of the fifth order maximum correspond to an angle θ_5'. Then,

$$\sin(\theta_5'+\alpha) - \sin\alpha = \sin\theta_5 = \frac{5\lambda}{d}.$$

Since λ/d is small, the difference between $\sin(\theta_5'+\alpha)$ and $\sin\alpha$ is also small, meaning θ_5' is a small angle too. We can then use $\sin(\theta_5'+\alpha) \approx \sin\alpha + \theta_5'\cos\alpha$ to get $\theta_5'\cos\alpha = \theta_5$, or $\theta_5' = \frac{5}{4}\theta_5$. From the problem statement we gather that

$$\Delta l = f(\tan\theta_5' - \tan\theta_5) \approx f(\theta_5' - \theta_5),$$

which implies $\Delta l = \frac{1}{4} f \theta_5$. However, we also have $\theta_5 = \frac{5\lambda}{d}$, so

$$\boxed{d = \frac{5\lambda f}{4\Delta l} = 50\,\mu\text{m}.}$$

(c) Only the beams travelling at less than a critical angle θ_{lim} will get focused on the screen. Tracking the rays passing through the centre of the lens, we find

$$\tan\theta_{\text{lim}} = \frac{L/2}{f} \quad \Rightarrow \quad \theta_{\text{lim}} = 5.7°.$$

After the rotation, angles θ_n corresponding to maxima obey

$$\sin(\theta_n + \alpha) - \sin\alpha = \frac{n\lambda}{d}.$$

The function on the left hand side increases with θ_n, which may vary between $-\theta_{\text{lim}}$ and $+\theta_{\text{lim}}$. Substituting these limits for θ, we find $n = -8.3$ and $n = 7.6$, respectively. This means that the maxima which fit on the screen start from $n = -8$ and end at $n = 7$. This is a total of $\boxed{N = 16}$ maxima. The number is still the same if we work with the exact values for α and d.

Experimental Exam

Problem 1. Oscillations of a wooden block.

Equipment:
Stand with pinch (to be used as a pivot for a pendulum), wire, wooden block, pliers, stopwatch, ruler, blank paper, graph paper.

The longest side of the block is denoted by a, the middle length one by b, and the shortest one by c. The mass of the block m is written down on the block itself. A hook with a screw thread can be inserted into holes on the surfaces of the block. This allows the block to oscillate about different axes. Denote the axes about which the block oscillates by [100] for the axis parallel to a, [010] for the axis parallel to b, and [001] for the axis parallel to c.

The aim of this problem is to calculate the acceleration due to gravity g **(10 pt)** and the torsion coefficient of the wire D **(5 pt)**. The two tasks are independent.

In order to find g you will need to study the period of rotational oscillations T of the pendulum about a horizontal axis which is perpendicular to one of the block's surfaces. The period is given by the formula

$$T = 2\pi\sqrt{\frac{I}{mgd\left(\frac{d}{l}+1\right)}},$$

where l is the length of the wire, d is the distance between the end of the wire and the centre of mass of the block, m is the mass of the block, and I is the moment of inertia of the block with respect to the axis of rotation. The moment of inertia is given by

$$I = \frac{1}{12}m(a_i^2 + a_j^2),$$

where a_i and a_j are the lengths of the edges perpendicular to the axis of rotation (i.e. a and b, or a and c, or b and c).

(a) Take enough useful measurements. Present them in a table and explain how they were obtained. **(4.5 pt)**

(b) State the variables which, when plotted, can easily give you g. **(0.5 pt)**

(c) Plot the relevant graph. **(3.0 pt)**

(d) Using the graph, determine the acceleration due to gravity g. **(1.0 pt)**

(e) Estimate your error in finding g. **(1.0 pt)**

In order to determine the torsion coefficient of the wire D you will need to study the period of rotational oscillations T of the pendulum about an axis coinciding with the wire. Their period is given by

$$T = 2\pi\sqrt{\frac{I}{D}},$$

where I is the moment of inertia about the axis of rotation (given above).

(f) Take enough useful measurements. Present them in a table and explain how they were obtained. **(2.25 pt)**

(g) State the variables which, when plotted, can easily give you D. **(0.25 pt)**

(h) Plot the relevant graph. **(1.75 pt)**

(i) Using the graph, determine the coefficient of torsion of the wire D. **(0.5 pt)**

(j) Estimate your error in finding D. **(0.25 pt)**

Call the examiner in case of any technical difficulties.

Note: Do not write on the block!