Short Exam 2

Problem. A monochromatic ($\lambda = 500\,\text{nm}$) point source S illuminates a rectangular mirror O which rotates at a frequency of $\nu = 16\,\text{Hz}$. The distance between the source and the mirror is $L = 100\,\text{m}$. The reflected light is sent to a detector D of negligible size located close to the source. The mirror rotates about an axis that lies in the plane of the mirror and is perpendicular to the plane in which the reflected ray moves. You can assume that the source, the detector, and the mirror are collinear. Accounting for diffraction, find the duration Δt of the light pulse registered by the detector. What is the width a of the mirror that minimises Δt? What is the value of Δt then? The height of the mirror is much larger than a.

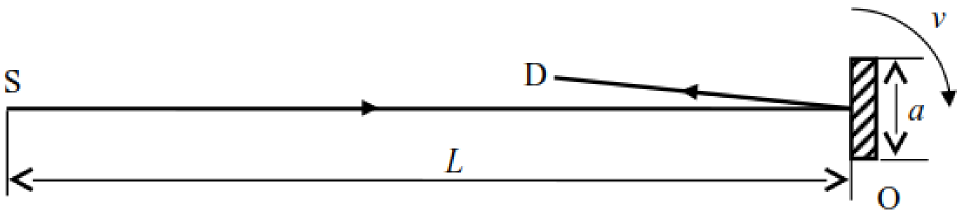
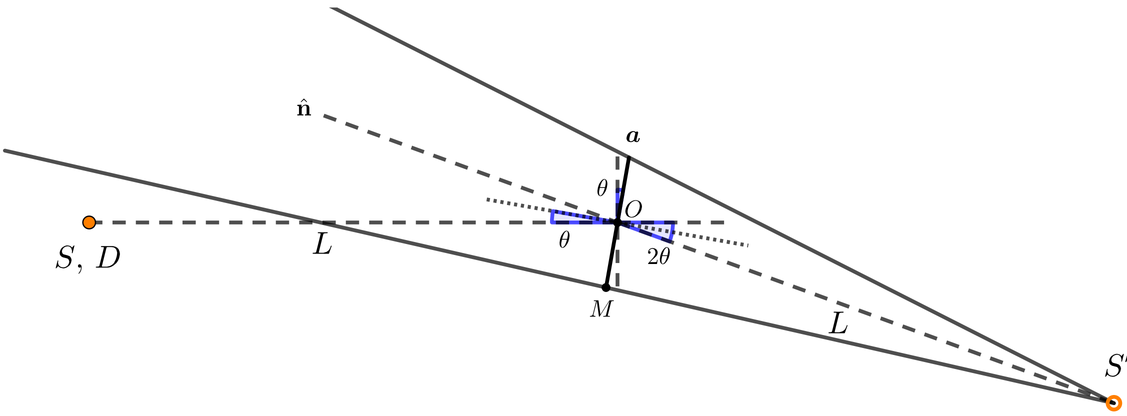


Figure 1

Solution. There are two factors to consider here. The first is related to ray optics. The mirror will get to rotate by some small angle θ_g starting from the position in the figure, after which no points on the mirror will reflect anything towards the detector[1].



Referring to the figure, this happens when the cone of the rays emanating from the image S' no longer includes the source S. This is the same as requiring $\angle OSS' \geq \angle OSM$. Given that $SS' \perp OM$, we have $\angle OSS' = \theta$, and since the mirror's tilt is small, we can write $\angle OSM = \frac{a/2}{L}$. Thus $\theta_g = \frac{a}{2L}$.

We'll also need to account for diffraction. The reflection from the mirror isn't perfect, and it will essentially act as an obstacle of size a which creates a diffraction pattern centered in the direction of the reflected ray $\hat{\mathbf{n}}$. The angular width of the central maximum of the pattern as seen from the mirror is $\frac{2\lambda}{a}$, with $\frac{\lambda}{a}$ on each side. The light within the maximum will land on

[1] In the original problem statement, the detector is said to be near the mirror rather than the source. But in that case you would definitely need the exact distance between the mirror and the detector, which isn't specified. I also have it on good authority that the answer for the optimal a is indeed $\sqrt{\lambda L}$, which can't be obtained unless the detector is actually near the source.

the detector even after the detector goes out of range for the geometrically reflected rays. This will cease after an additional rotation of θ_d, such that $\angle SON = 2\theta_d = \frac{\lambda}{a}$. In total, the mirror rotates by $\theta_g + \theta_d = \frac{a}{2L} + \frac{\lambda}{2a}$ until the end of the pulse. The total length of the pulse corresponds to twice that value as the detector is accessible from before the mirror is vertical. Thus,

$$\Delta t = \frac{2(\theta_g + \theta_d)}{\omega} = \boxed{\frac{(a/L) + (\lambda/a)}{2\pi\nu}.}$$

To find the minimum of Δt, we set the derivative with respect to a to zero, whereby

$$\frac{1}{L} - \frac{\lambda}{a^2} = 0 \quad \Rightarrow \quad \boxed{a = \sqrt{\lambda L} = 7.1\,\text{mm}.}$$

Returning to the expression for Δt, we have

$$\boxed{\Delta t = \frac{1}{\pi\nu}\sqrt{\frac{\lambda}{L}} = 1.4 \times 10^{-6}\,\text{s}.}$$

Theoretical Exam

Problem 4. Figure 2 shows the I-V curve of a bulb. This bulb is connected in parallel with a resistance $R = 2.00\,\Omega$ to a source of EMF $E = 15\,\text{V}$ and internal resistance $r = 3.00\,\Omega$. Using the I-V curve, find the power P dissipated in the bulb. Estimate the accuracy ΔP of your result.

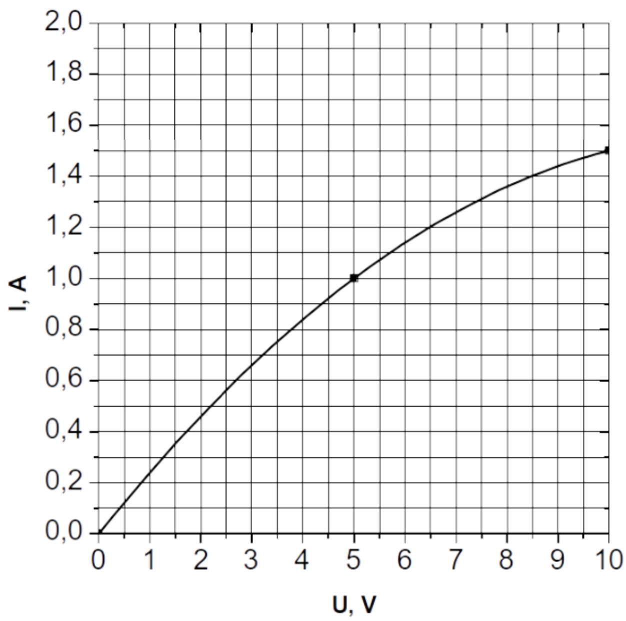


Figure 2

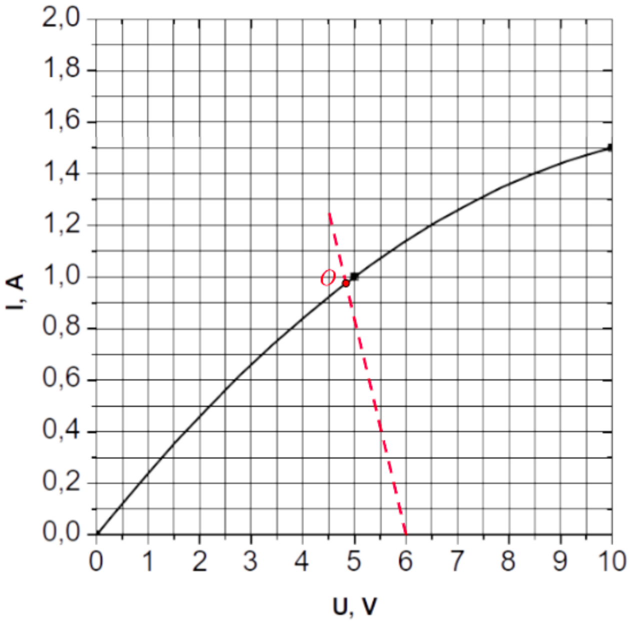
Solution. Using Kirchhoff's rules, we will find a constraint on the state of the lightbulb (U, I). Firstly, note that the voltage across the resistance R is also U, and hence the current through it is U/R. Then, the total current through r is $I + \frac{U}{R}$, and so the loop rule gives us

$$E - \left(I + \frac{U}{R}\right) r - U = 0 \quad \Rightarrow \quad I = \frac{E}{r} - \frac{R+r}{Rr}U \quad \Rightarrow \quad I = 5\,\text{A} - (0.833\,\Omega^{-1})\,U.$$

The voltage and the current of the lightbulb correspond to the point on the I-V curve which follows the constraint. On the graph, this constraint looks like a straight line. After we plot it and intersect it with the I-V curve, we estimate $U = 4.8\,\mathrm{V}$ and $I = 0.98\,\mathrm{A}$. Our results are very sensitive to the slope of the line, so we should have generous error estimates, e.g. $\Delta U = 0.1\,\mathrm{V}$ and $\Delta I = 0.02\,\mathrm{A}$. The dissipated power is $P = UI = 4.704\,\mathrm{W}$, while the error in P is

$$\Delta P = \Delta(UI) = \Delta U I + U \Delta I = 0.194\,\mathrm{W}.$$

To present our answer properly, we need to round the error to one significant digit, and work with the same precision for P. Thus, $\boxed{P = (4.7 \pm 0.2)\,\mathrm{W}}$.



Problem 5. A coil of inductance $L = 2.0\,\mu\mathrm{H}$ and internal resistance $r = 1.0\,\Omega$ is connected in parallel to a resistance $R = 2.0\,\Omega$. These have been connected to a constant voltage source $E = 3.0\,\mathrm{V}$ for a long time. At time $t = 0$ the source is removed from the rest of the circuit. Find the time dependence of the current through the coil $I(t)$. Find the total heat Q dissipated in the coil until the current ceases to flow.

Solution. The coil in this problem can be modelled as an ideal inductance L connected in series with a resistance r. Before the source is removed, the circuit is in a steady state – the currents through the components are constant, and there's no back EMF from the inductance. More specifically, the currents through the inductor and the resistor are $I_1 = E/r$ and $I_2 = E/R$, respectively. When the source is disconnected, the inductor doesn't allow any sudden change in the current through it. Since the back EMF is given by $U_L = -L\frac{\mathrm{d}I}{\mathrm{d}t}$, any significant jump in I within the instant $\mathrm{d}t$ when we're disconnecting the source will result in U_L diverging, which is unphysical. It follows that the initial current (with no source) is $I_1 = E/r$ along the whole loop of L, r, and R.

Now, the loop equation is

$$\left(-L\frac{\mathrm{d}I}{\mathrm{d}t}\right) - Ir - IR = 0,$$

which has the solution

$$I = Ae^{-\frac{R+r}{L}t}, \quad I(0) = I_1 = \frac{E}{r} \quad \Rightarrow \quad \boxed{I(t) = \left(\frac{E}{r}\right) e^{-\frac{R+r}{L}t}.}$$

The heat dissipated at r (i.e. at the coil) is then

$$Q = \int_0^\infty I^2 r \,\mathrm{d}t = \boxed{\frac{E^2 L}{2(R+r)r} = 3\,\mu\mathrm{J}.}$$

Problem 6. A parallel beam of monochromatic light with wavelength $\lambda = 589.0\,\text{nm}$ is normally incident on a grating of period $d = 2.5\,\mu\text{m}$ and $N = 10000$ slits. Find the angular width of the diffraction maximum of order $m = 2$. Derive the relevant formula and calculate the numerical value in arcminutes.

Solution. This is essentially a problem about N-slit interference, where will we have to derive an expression for the intensity pattern. We will use complex numbers to avoid dealing with cumbersome trigonometric identities. For diffraction at an angle θ from the normal, the waves due to any two adjacent slits will have a phase difference of $\Delta\varphi = \frac{2\pi}{\lambda} d \sin\theta$. After we assign an amplitude A to each slit, the superposition of all the waves can be found as the sum of a geometric series:

$$\left(A + Ae^{i\Delta\varphi} + Ae^{2i\Delta\varphi} + \cdots + Ae^{(N-1)i\Delta\varphi}\right) e^{-i\omega t} = Ae^{-i\omega t} \cdot \frac{1 - e^{iN\Delta\varphi}}{1 - e^{i\Delta\varphi}}.$$

This corresponds to a wave of amplitude

$$A' = A\left|\frac{1 - e^{iN\Delta\varphi}}{1 - e^{i\Delta\varphi}}\right| = A\frac{\left|1 - e^{iN\Delta\varphi}\right|}{\left|1 - e^{i\Delta\varphi}\right|} = A\frac{\left|(1 - \cos N\Delta\varphi) - i(\sin N\Delta\varphi)\right|}{\left|(1 - \cos \Delta\varphi) - i(\sin \Delta\varphi)\right|}.$$

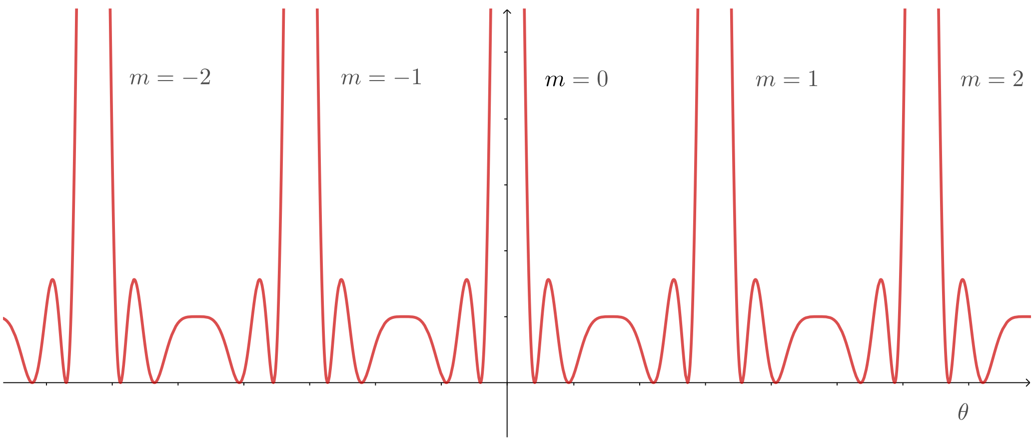
We know from the theory of waves that the intensity is proportional to the square of the amplitude. Then, the intensity $I(\theta)$ from the grating relates to the intensity I_0 from a single slit like

$$I(\theta) = I_0\left(\frac{A'}{A}\right)^2 = I_0\frac{(1 - \cos N\Delta\varphi)^2 + (\sin N\Delta\varphi)^2}{(1 - \cos \Delta\varphi)^2 + (\sin \Delta\varphi)^2} = I_0\frac{2 - 2\cos N\Delta\varphi}{2 - 2\cos \Delta\varphi} = I_0\frac{\sin^2\left(\frac{N\Delta\varphi}{2}\right)}{\sin^2\left(\frac{\Delta\varphi}{2}\right)}.$$

Now to interpret this expression. Both the numerator and the denominator cycle between 0 and 1 as the angle θ increases, but the numerator does so much more rapidly. It seems that the maxima will be observed when the denominator is small. Indeed, in the limiting case when both the sines are small, we have

$$I(\theta) = I_0\frac{\left(\frac{N\Delta\varphi}{2}\right)^2}{\left(\frac{\Delta\varphi}{2}\right)^2} = N^2 I_0,$$

Thinking again in terms of wave superposition, this corresponds to the case where all the waves are in phase, so that $A' = NA$. This requires $\Delta\varphi = 2\pi m$, $m \in \mathbb{Z}$. It it those m that we use to indicate the order of the maxima. We should mention that there are also additional subsidiary maxima which occur every time the numerator in the general expression for $I(\theta)$ reaches unity, no matter the value of the denominator. These, however, are negligible in the case of an ideal grating ($N \to \infty$), so they're not part of the convention for labelling maxima. When we say $m = 2$, we mean the second-order primary maximum where $\Delta\varphi = 4\pi$ and hence $\sin\theta = \frac{2\lambda}{d}$. To illustrate all this, here's the intensity pattern for $N = 5$:



Let's proceed with finding the width of the $m = 2$ maximum. You'll need to know that 'width' means the distance between the two minima on either side of the maximum. These minima correspond to the instances of $\sin^2\left(\frac{N\Delta\varphi}{2}\right) = 0$ which lie nearest to $\Delta\varphi = 4\pi$. In that case $\Delta\varphi = 4\pi \pm \frac{2\pi}{N}$, or $\sin\theta = \frac{2\lambda}{d} \pm \frac{\lambda}{Nd}$. These values for θ, which we'll denote by θ_+ and θ_-, are very close. We seek their difference $(\theta_+ - \theta_-) \equiv \Delta\theta$. Subtracting the sines, we find

$$\sin\theta_+ - \sin\theta_- = \frac{2\lambda}{Nd} = 2\sin\left(\frac{\Delta\theta}{2}\right)\cos\left(\frac{\theta_+ + \theta_-}{2}\right) = \Delta\theta\cos\theta.$$

Here $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{m\lambda}{d}\right)^2}$, so

$$\boxed{\Delta\theta = \frac{2\lambda}{N\sqrt{d^2 - (m\lambda)^2}} = 0.184'.}$$

Experimental Exam

Problem 1. The speed of sound in air.

Equipment:
Tone generator, two loudspeakers, microphone with a jack (female XLR connector), rectifier, resistance, two-channel oscilloscope, wire connectors, wires, screwdriver, self-retracting tape measure, tape, graph paper.

(a) Connect the setup to the microphone jack. The jack has three pins called base (−), middle (+), and top (signal). The middle and top pins are shorted. Carefully examine the jack. Using the wire connectors and the wires, connect the jack and the resistance in series with the rectifier. Also connect the oscilloscope to the microphone so as to measure the voltage across it. Sketch the circuit. Supply a voltage of 3 V to the microphone and test it. **(1.5 pt)**

Note: To avoid damaging the microphone when turning it on, follow these instructions:
- Before connecting the microphone to the rectifier, set the voltage to zero using the potentiometers.
- First, turn on the rectifier.
- Then connect the wires to the rectifier's terminals, being mindful of the polarity.
- **Slowly** increase the voltage to 3 V.

Note: To avoid damaging the microphone when turning it off, do not turn off the rectifier while the microphone is connected. First turn down the voltage to zero, detach the wires and only then turn off the rectifier.

(b) Connect the two loudspeakers to the tone generator. Put the microphone very close to one of the loudspeakers. Vary the frequency of the tone generator in the range [2 kHz, 20 kHz]. Write down the frequency at which the voltage across the microphone is largest. Work with this frequency from now on. **(1 pt)**

Design a setup for observing two-source interference. The distance d between the loudspeakers should be around 30 cm to 50 cm. The distance L between the line through the loudspeakers and the line along which you will move the microphone should be around 1 m. Put the loudspeakers and the microphone on separate tables. Remove all objects that could reflect the sound waves. Be mindful of where you stand during the measurements.

(c) Move the microphone around and measure the coordinates x_k of at least 6 consecutive minima around the central maximum. Use a voltage low enough so that the loudspeakers emit monochromatic waves (without any harmonics). Write down the value of this voltage. Write down the values of x_k in a table. Use those to find an accurate value for the coordinate of the central maximum x_0. **(3 pt)**

(d) Derive an exact formula for the optical path difference Δ of the interfering sound waves. Express Δ in terms of L, d, x, and x_0. **(1 pt)**

(e) Plot a graph of the optical path difference Δ for the measured minima against some number that corresponds to the physical condition for observing such minima. **(1 pt)**

(f) Find the wavelength of the sound waves λ. **(1 pt)**

(g) Calculate the speed of sound in air c for your experimental setup. **(0.5 pt)**

(h) Estimate the error in your value for c. **(1 pt)**